that case the function f_n represents the joint density, g_n the marginal density. The point x_n represents the center of the marginal distribution g_n and $y_n = y_n(x)$ is the conditional mode of the distribution of y given x. Of course this approximate marginalization does not require f_n to be a posterior distribution. Phillips (1983) uses this approach to approximate the marginal sampling distribution of various econometric estimators.

The asymptotic properties of the saddlepoint approximation to sampling distributions and the Laplace approximation to marginal densities are very similar. Both yield approximations that have errors uniformly of order $O(n^{-1})$ on fixed neighborhoods of x_n. Both benefit from numerical renormalization in the sense that the absolute errors of the approximations are of order $O(n^{-3/2})$ in $n^{-1/2}$-neighborhoods of x_n. Another interpretation of this result is that the shapes of the densities $g_n(\cdot)$ are approximated to order $O(n^{-3/2})$ by both methods.

The approximation of posterior expectations by Laplace's method is somewhat different. A single number is to be approximated rather than a function. Direct application of Laplace's method yields the maximum likelihood estimate or the posterior mode as an approximation to the posterior mean. The error of this approximation is of order $O(n^{-1})$. More accurate approximations with an error of order $O(n^{-2})$ can be obtained by using higher order terms, as described by Lindley (1980), or by using different centers for the expansions of numerator and denominator integrals, as described in Tierney and Kadane (1986) and Tierney, Kass and Kadane (1987).

The approximate predictive densities discussed in Leonard (1982), Tierney and Kadane (1986) and Davison (1986) fall somewhere between marginal density and moment approximations. Because a predictive density is a density, its approximation would appear to be more closely related to the approximation of marginal and sampling densities. On the other hand, the predictive density at a particular point can be expressed as a posterior expectation. The result of applying second order expectation approximation, as in (4.1) of Tierney and Kadane (1986), is an approximation to the predictive density with an error of order $O(n^{-2})$. The order of this error term will generally not be improved by numerical renormalization. As a result I feel that these approximate predictive densities are more closely related to approximate expectations than to approximate marginal densities.

I hope that these comments have added to the discussion in Section 6 of Professor Reid's excellent paper.

ADDITIONAL REFERENCES

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Comment

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The world of asymptotics is beautiful and mysterious. Witness Stirling's approximation, and recall the first time you needed to use it. What explains the odd yet simple formula, you may have asked, and more, How is it that with one correction term it already achieves 99.95% accuracy in approximating factorials as small as 2? Marvel at Figure 1. But recognize, each time we consider a sample of size n to be part of an infinite sequence of observations, we are faced with an irony: limits do not depend on the first n values, yet they are able to inform us about the behavior of the sample. Our finite world seems tied to asymptopia, but how?

Second-order asymptotic results continue to produce this feeling of awe and amazement in those who aren't yet familiar with them. Nancy Reid's review not only tells the saddlepoint story, it also nicely demonstrates the similarity of method in applications to maximum likelihood and conditional inference, robust estimation and Bayesian analysis. My comment consists of (i) a brief description of the relationship between Laplace's method and the saddlepoint

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