Comment

Minoru Sakaguchi

Professor Ferguson provides a nice introduction, both historical and mathematical, to the now "classical" secretary problem (CSP). He then refers to the game of googol as a striking example of the "true" secretary problem according to his definition, and tries to find the solution which he calls "a resolution" to the problem. In doing so he discusses a problem of information and sequential decision-making in a competitive two-person game with a very simple payoff structure. My purpose here is (a) to mention that Cayley's problem in 1875 already closely resembled the CSP which appeared 80 years later, (b) to notice that choosing the second best is more difficult than choosing the best, and (c) to note that the game of googol involves various classes of games according to the minimizing player's behavior in restricting his set of choices.

CAYLEY'S PROBLEM AND THE CSP

Cayley's problem in its first proposed form can also be resurrected from oblivion in the following way:

An urn contains n balls, each one labeled with one of the distinct positive integers $1, 2, \cdots, n$. One ball at a time is randomly drawn, the number on it read, and the ball removed. You may continue this procedure at most k times, each time one ball being removed. You are allowed to terminate the procedure at your convenience. If you observe the numbers on the balls $x_1, x_2, \cdots, x_{m-1}$ and stop at the mth drawing with the number x_m, then you win the game if the future draws at times $m+1, m+2, \cdots, k$ yield the result $x_m > \max\{x_1, \cdots, x_{m-1}, X_{m+1}, \cdots, X_k\}$. The problem is to find a stopping strategy that maximizes the probability of your winning.

This problem with $1 \le k \le n$ evidently (maybe) is a true secretary problem according to Ferguson's definition of it, but finding an optimal stopping strategy is far more complicated than in Cayley's original problem or in CSP. The reason I attached "maybe" above is that the absolute (as well as relative) ranks of the balls are known as they are drawn.

Define the state (m, y), for $m = 1, 2, \cdots, k$; $1 \le y \le n$, to mean that (1) you have not yet stopped the procedure and (2) you face the mth draw with the number $X_m = y$ which is a "candidate" (i.e., it is larger than any of the numbers which have been observed in the past $(m-1)$ draws). Then in state (m, y) the urn contains balls with numbers $(\{1, 2, \cdots, y\}\setminus\{x_1, \cdots, x_{m-1}, y\}) \cup \{y+1, \cdots, n\}$ and therefore the transition probability from state (m, y) is

$$\Pr\{(m', z) \mid (m, y)\} = \begin{cases} 0, & \text{if } z<y, \\ \dfrac{y-m}{n-m}\cdot\dfrac{y-m-1}{n-m-1}\cdots\dfrac{y-m'+2}{n-m'+2}\cdot\dfrac{1}{n-m'+1}, & \text{if } y+1 \le z \le n, \end{cases}$$

for $m < m' \le k$. Letting $V_{m,k}(y)$ denote the maximum expected probability of winning when the state of the process is (m, y), the principle of dynamic programming yields the equation

$$\begin{aligned} V_{m,k}(y) &= \max\Bigg[\frac{y-m}{n-m}\cdot\frac{y-m-1}{n-m-1}\cdots\frac{y-k+1}{n-k+1}, \\ &\qquad \sum_{m<m'\le k}\ \sum_{y+1\le z\le n} \Pr\{(m', z)\mid(m, y)\} V_{m',k}(z)\Bigg] \\ &= \max\Bigg[\frac{(y-m)_{k-m}}{(n-m)_{k-m}}, \sum_{m<m'\le k}\frac{(y-m)_{m'-m-1}}{(n-m)_{m'-m-1}} \\ &\qquad \cdot\frac{1}{n-m'+1}\sum_{y+1\le z\le n} V_{m',k}(z)\Bigg], \\ &\qquad 1 \le m \le k;\ V_{k,k}(y) \equiv 1, \end{aligned} \tag{1}$$

where $(j)_i$, with $1 \le i \le j$, means $j(j-1)\cdots(j-i+1)$.

For $n = 4$, as in Cayley's example, we find by direct computation without using the above Optimality Equation (1) that the probability of winning under an optimal stopping strategy is

$$W_k^{(4)} \equiv \frac{1}{4}\sum_{y=1}^{4} V_{1,k}(y) = 1, \frac{5}{6}, \frac{5}{6}, 1, \quad \text{for } k = 1, 2, 3, 4.$$

Going to the limit where $k, n \to \infty$, but with k/n tending to a fixed $\alpha \in (0, 1)$, dilutes the difference between sampling with and without replacement and therefore we conjecture that $W_k^{(n)}$ tends to the Gilbert-Mosteller constant 0.58106,

Minoru Sakaguchi is Professor of Engineering Sciences at Osaka University. His mailing address is: Faculty of Engineering Sciences, Osaka University, Toyonaka, Osaka, Japan.