Statistical Science
1990, Vol. 5, No. 3, 356–369

Stein Estimation: The Spherically Symmetric Case

Ann Cohen Brandwein and William E. Strawderman

Abstract. This paper presents an expository development of Stein estimation with substantial emphasis on exact results for spherically symmetric distributions. The themes of the paper are: a) that the improvement possible over the best invariant estimator via shrinkage estimation is not surprising but expected from a variety of perspectives; b) that the amount of shrinkage allowable to preserve domination over the best invariant estimator is, when properly interpreted, relatively free from the assumption of normality; and c) that the potential savings in risk are substantial when accompanied by good quality prior information.

Key words and phrases: Stein estimation, decision theory, minimaxity, quadratic loss, concave loss, spherical symmetry.

1. INTRODUCTION

This paper presents an expository development of Stein estimation with substantial emphasis on exact results for nonnormal location models. The themes of the paper are: a) that the improvement possible over the best invariant estimator via shrinkage estimation is not surprising but expected from a variety of perspectives; b) that the amount of shrinkage allowable to preserve domination over the best invariant estimator is, when properly interpreted, relatively free from the assumption of normality; and c) that the potential savings in risk are substantial when accompanied by good quality prior information.

Relatively, much less emphasis is placed on choosing a particular shrinkage estimator than on demonstrating that shrinkage should produce worthwhile gains in problems where the error distribution is spherically symmetric. Additionally such gains are relatively robust with respect to assumptions concerning distribution and loss.

The basic problem, of course, is the estimation of the mean vector θ of a p-variate location parameter family. In the normal case (with identity covariance) for $p = 1$, the usual estimator, the sample mean, is the maximum likelihood estimator, the UMVUE, the best equivariant and minimax estimator for nearly arbitrary symmetric loss, and is admissible for essentially arbitrary symmetric loss. Admissibility for quadratic loss was first proved by Hodges and Lehmann (1950) and Girshick and Savage (1951) using the Cramér–Rao inequality and by Blyth (1951) using a limit of Bayes type argument.

For $p = 2$, the above properties also hold in the normal case. Stein (1956) proved admissibility using an information inequality argument. In that same paper, however, Stein proved a result that astonished many and which has led to an enormous and rich literature of substantial importance in statistical theory and practice.

Stein (1956) showed that estimators of the form $(1 - a/(b + \|X\|^2))X$ dominate X for a sufficiently small and b sufficiently large when $p \geq 3$. James and Stein (1961) sharpened the result and gave an explicit class of dominating estimators, $(1 - a/\|X\|^2)X$ for $0 < a < 2(p - 2)$. They also indicated that a version of the result holds for general location equivariant estimators with finite fourth moment and for loss functions which are concave functions of squared error loss. Brown (1966) showed that inadmissibility of the best equivariant estimator of location holds for virtually all problems for $p \geq 3$, and, in Brown (1965), that admissibility tends to hold for $p = 2$. Minimaxity for all p follows from Kiefer (1957).

Section 2 gives a geometrical argument due to Stein which indicates that shrinkage might be expected to work under quite broad distributional assumptions.

Ann Cohen Brandwein is Associate Professor of Statistics at Baruch College. Her mailing address is Department of Statistics, Baruch College, Box 513, 17 Lexington Avenue, New York, New York 10010. William E. Strawderman is Professor and Chairman of Statistics at Rutgers University. His mailing address is Department of Statistics, Rutgers University, New Brunswick, New Jersey 08903.