REJOINDER

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The basic elements of our paper, volume testing and components of variance, are staples of good practice in the application of ordinary linear models. Table A gives a simple example. Twenty dice have been seized from a fictitious Las Vegas casino, on the suspicion that the casino has subtly overweighted the occurrence of "1" and "6." The police roll each die 50,000 times, obtaining the results shown. Is the casino guilty?

Let m_i be the number of occurrences observed for the ith die. The total $m = \sum_{i=1}^{20} m_i$ equals 335,294. If the dice are fair, m is nearly normally distributed with mean 333,333.3 and standard deviation $[10^6(1/3)(2/3)]^{1/2} = 471.4$. The obvious normal test statistic $z = [335{,}294 - 333{,}333.3]/471.4$ equals 4.16, significance level $2 \cdot 10^{-5}$, overwhelming proof against the hypothesis of fairness. The multidimensional test statistic

$$\chi^2 = \sum_{i=1}^{20} (m_i - 16{,}666.7)^2/[50{,}000(1/3)(2/3)] = 687.9$$

is also overwhelmingly significant when compared to the null distribution χ^2_{20}. On the other hand, the t-statistic

$$t = (\bar{m} - 16{,}666.7)/[\Sigma(m_i - \bar{m})^2/(19 \cdot 20)]^{1/2} = 0.70$$

is not at all significant, with attained level only .246 compared to a t_{19} distribution. Which test should we believe?

The correct answer is the t-test. The dice are unfair, as the χ^2 test shows, but not in the systematic manner of which the casino was accused. The trouble with the z-test, which is just the t-test applied to all 1,000,000 rolls, is that the sample size in this problem is really 20 and not 1,000,000. This is clear from a components of variance analysis of the m_i, which indicates that the component due to variation between dice is about 34 times larger than the binomial variation from 50,000 rolls.

All of this is standard statistical practice. The point of bringing it up here is that the t-test, which leads to the correct conclusion, is in fact a volume test. The centered, renormalized data vector $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, where $u_i = (m_i - 16{,}666.7)/[\sum_{j=1}^{20} (m_j - 16{,}666.7)^2]^{1/2}$, is a point on $\mathcal{SP}_{20}$, the unit sphere in 20 dimensions. The set of vectors $\mathbf{U}$ lying as close or closer than $\mathbf{u}$ to the point $\mathbf{e} = (1, 1, \cdots, 1)/\sqrt{20}$ is a spherical cap on $\mathcal{SP}_{20}$, centered at $\mathbf{e}$. The significance

TABLE A

Number of occurrences of either 1 *or* 6 *in* 50,000 *rolls each of twenty dice; total number of occurrences is* 335,294 *out of* 1,000,000 *total rolls. Were the dice weighted?*

17755	16734	16769	16359	16661	16285	16309
17479	16529	16486	15668	16292	17511	17020
15929	16829	17665	17981	16677	16356	