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WEAK CONVERGENCE OF THE EMPIRIC PROCESS FOR INDEPENDENT RANDOM VARIABLES

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Introduction. This paper investigates the (weak) convergence properties of the Empiric Process in a more general framework of independent random variables (without common distribution). In this situation the Empiric Process converges to a Gaussian Process on the unit interval which is "dominated" by the Brownian Bridge.

The weak convergence of the Empirical Process, the "smoothed" Empirical Process and the Empiric Process for a sequence of independent random variables is derived without the restriction that they have a common distribution. The characteristics of the Gaussian Process which is their "weak" limit is discussed.

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1. Theoretical preliminaries. Let S be a metric space, $\mathscr{S}$ the σ-algebra generated by the open sets. If P_n and P are probability measures on $(S, \mathscr{S})$ such that $\int_S f\,dP_n \to \int_S f\,dP$ for every bounded continuous function f on S, we say that P_n converges weakly to P and write $P_n \Rightarrow P$. Further generic properties of weak convergence can be found in [2]; in particular, $P_n \Rightarrow P$ if and only if $P_n(A) \to P(A)$ for all subsets A such that $P(\partial A) = 0$ where ∂A is the boundary of A.

Let Y_n and Y be random elements of S. We say that Y_n converges in distribution to Y and write $Y_n \to_{\mathscr{D}} Y$ if and only if the probability distributions of Y_n converge weakly to the probability distribution of Y.

For the case of $\hat{S}(x, \omega)$, the empirical distribution function of independent identically distributed random variables with continuous distribution F, weak convergence can be used to prove the Glivenko–Cantelli Theorem ([3], page 20)

$$P[\omega \mid \sup_{-\infty<x<\infty} |\hat{S}_n(x, \omega) - F(x)| \to 0] = 1 \tag{1}$$

and the sharper result due to Kolmogorov ([2], page 104)

$$P[\omega \mid n^{\frac{1}{2}} \sup_{-\infty<x<\infty} |\hat{S}_n(x, \omega) - F(x)| \leqq \alpha] \to 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} e^{-2k^2\alpha^2} \tag{2}$$

for all $\alpha \geqq 0$. The method of proof involves weak convergence of random elements of the metric space D, of all real-valued functions on $[0, 1]$ which are right continuous and have left-hand limits. The metric used for D is defined by ([2], page 112)

$$d_0(x, y) = \inf\{\varepsilon > 0 \mid \|\lambda\| \leqq \varepsilon, \sup_{0 \leqq t \leqq 1} |x(t) - y(\lambda(t))| \leqq \varepsilon\}$$

where $x, y \in D$, λ is a non-decreasing function on $[0, 1]$ such that $\lambda(0) = 0$ and

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