

THE 1973 WALD MEMORIAL LECTURES

THE PROBABILITY THEORY OF ADDITIVE
ARITHMETIC FUNCTIONS^{1,2}

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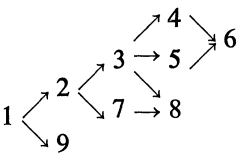
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Probability limit theorems for additive functions of positive integers
are reviewed.

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A review of probability methods in multiplicative arithmetic, with liberal use of probabilistic language, reasoning, and technique, and sparing use of the apparatus of number theory, is the object of this paper. Except for the fundamental theorem of arithmetic and its immediate consequences, what number theory is required is developed in Section 10. As for probability, the ordinary limit theory for random variables is enough except for the weak-convergence results in Sections 4 and 6 and the entropy calculations in Section 9. The diagram below shows the logical interdependence of the first nine sections; the remaining three can be consulted as the need arises.



1. Introduction. *Additive functions.* An arithmetic function f , defined over the positive integers, is *additive* if $f(m_1 m_2) = f(m_1) + f(m_2)$ whenever m_1 and m_2

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