The Annals of Probability
2002, Vol. 30, No. 1, 474–479

BOOK REVIEW

PAUL MALLIAVIN, *Stochastic Analysis*. Springer, New York, 1997, 370 pages, \$125.00.

REVIEW BY DENIS BELL

University of North Florida

This book is an exposition of some important topics in stochastic analysis and stochastic geometry. In reviewing the book, it is as well to start with Malliavin's main contribution to the field.

The *Malliavin calculus*, introduced in the mid-seventies in the papers [9] and [10], was motivated by the following link between stochastic analysis and partial differential equations: consider the Stratonovich stochastic differential equation

$$dx_t = \sum_{i=1}^{n} X_i(x_t) \circ dw_i(t) + X_0(x_t)\,dt \tag{1}$$

where $X_0, \dots, X_n$ are smooth vector fields defined on $\mathbb{R}^d$ and $w = (w_1, \dots, w_n)$ is a standard Wiener process. Let L denote the second-order differential operator

$$\tfrac{1}{2}\sum_{i=1}^{n} X_i^2 + X_0.$$

It has been known since the early work of Itô that the solution process x_t in (1) is Markov and that its transition probabilities yield the fundamental solution (in the sense of distributions) to the heat equation

$$\frac{\partial u}{\partial t} = Lu.$$

Suppose now that the vector fields $X_1, \dots, X_n$, together with all the Lie brackets generated by $X_0, \dots, X_n$, span $\mathbb{R}^d$ at every point [we'll refer to this assumption as (HC)]. Then an application of Hörmander's hypoellipticity theorem implies that the transition probabilities $p(t, x, dy)$ of x_t admit smooth densities $p(t, x, y)$, for all positive t. Malliavin reversed this flow of information from PDE theory to probability by proving *directly* that condition (HC) implies the existence of smooth densities for the process x_t. From here standard techniques can be used to deduce that the operator $\partial/\partial t - L$ is hypoelliptic, thus yielding a probabilistic proof of Hörmander's theorem.

Received April 2001.