INEQUALITIES AMONG AVERAGES

By Nilan Norris

Numerous inequalities among averages of various types are condensed in the monotonic character of the function

$$\phi(t) = \left(\frac{x_1^t + x_2^t + \cdots + x_n^t}{n}\right)^{\frac{1}{t}}$$

of the positive numbers $x_1, x_2, \cdots, x_n$, not all equal each to each. For $t = -1$ this function is the harmonic mean; for $t = 0$ it is the geometric mean; for $t = 1$ the arithmetic mean; and for $t = 2$ the root mean square. The relations among these four means which customarily are proved by special and disconnected methods appear easily as applications of the theorem that $\phi(t)$ is an increasing function of t. That is, for any values of t_1 and t_2 such that $-\infty < t_1 < t_2 < +\infty$, it will be true that $\phi(t_1) < \phi(t_2)$. Several proofs of this theorem have been published, many of them very complex. An extremely simple proof is herewith presented.[1]

That $\phi(t)$, $\phi'(t)$ and $\phi''(t)$ all exist and are continuous for all real values of t may be shown by expanding each of the quantities x_i^t in a series of powers of t and considering the remainders after each of the first three terms. The ordinary rule for evaluating forms reducing to 0/0, which requires the function under consideration to be continuous and to have at least a continuous first derivative for $t = 0$, may then be applied to $[\log \phi(t)]/t$ to show that $\phi(0)$ is the geometric mean. It is clear that $\phi(-\infty)$ and $\phi(+\infty)$ are respectively the least and the greatest of the x_i. This fact and the monotonic property of $\phi(t)$ make it evident that for each real value of t, the function may be regarded as an average in the usual sense that it lies within the range of the observations.

For a simple demonstration of the increasing character of $\phi(t)$, consider the auxiliary function

$$F(t) = t^2 \frac{\phi'(t)}{\phi(t)} = t^2 \frac{d}{dt}\left\{\frac{1}{t} \log \frac{\Sigma x^t}{n}\right\} = t\,\frac{\Sigma x^t \log x}{\Sigma x^t} - \log \frac{\Sigma x^t}{n}.$$

It is clear that $\phi'(t)$ has the same sign as $F(t)$. The theorem will be proved by showing that the sign of $F(t)$ is positive for all values of t except zero, when $\phi'(t)$ vanishes.

---

[1] Professor Harold Hotelling rendered invaluable assistance in condensing for publication the material herein presented from a more extended study of generalized mean value functions.