# ON THE FREQUENCY FUNCTION OF $xy$

By Cecil C. Craig

Given the distribution function of $x$ and $y$, what can be said of the distribution of the product $xy$? The author has had two inquiries during the last two years, one from an investigator in business statistics and the other from a psychologist, concerning the probable error of the product of two quantities, each of known probable error. There seems to be very little in the literature of mathematical statistics on this question.

If $x$ and $y$ are independent and are each distributed according to the same normal frequency law, it is well known that the distribution function of

$$\bar{z} = \frac{x - m_x}{\sigma_x} \cdot \frac{y - m_y}{\sigma_y}$$

is

$$\frac{1}{\pi} K_0(\bar{z}) \text{ ,}$$[1]

in which $K_0(\bar{z})$ is the Bessel function of the second kind of a purely imaginary argument of zero order.[2] If $x$ and $y$ are independent and are each distributed according to a logarithmic normal frequency law, it has been pointed out that the product, $(x - a)$ $(y - b)$, in which $a$ and $b$ are the upper (or lower) limits of the range for $x$ and $y$ respectively, is distributed according to a law of the same type.[3] In both cases the special choice of origins greatly simplifies the problem.

In the present discussion it will be assumed that $x$ and $y$ are distributed normally. It will appear that the distribution of $xy$ is a function of $r_{xy}$, the coefficient of correlation between $x$ and $y$, and of the parameters,

$$\rho_1 = \frac{m_1}{\sigma_1} = \frac{m_x}{\sigma_x} \quad \text{and} \quad \rho_2 = \frac{m_2}{\sigma_2} = \frac{m_y}{\sigma_y} \text{ ,}$$

which are proportional to the reciprocals of the coefficients of variation. The chief difficulty arises when $\rho_1$ and $\rho_2$ are small so that zero values of $xy$ occur

[1] J. Wishart and M. S. Bartlett: The Distribution of Second Order Moment Statistics in a Normal System; Proceedings of the Cambridge Philosophical Society, Vol. XXVIII (1932), pp. 455–459.

[2] G. N. Watson: A Treatise on the Theory of Bessel Functions; Cambridge University Press (1922), p. 78.

[3] P. T. Yuan: On the Logarithmic Frequency Distribution and the Semi-logarithmic Frequency Surface; Annals of Mathematical Statistics, Vol. 4 (1933), pp. 46, 47.