NOTE ON NUMERICAL EVALUATION OF DOUBLE SERIES[1]

1. The Euler-Maclaurin summation formula has been extended to two variables by Dr. Sheppard,[2] and Mr. Irwin,[3] to determine cubature formulas. A more complicated two-dimensional form was given by Baten[4] involving product polynomials, for which a remainder term was also calculated. The purpose of this note is to apply the simpler formula to the numerical evaluation of double series of positive terms. The method may be extended to multiple series of order $p > 2$. If the double series converges one may sum by rows (or columns), using the ordinary sum formula twice. The method is to take out a rectangular block of mn terms and then apply the formula to the remaining terms. By taking m and n sufficiently large one may cause the series resulting from the formula to converge sufficiently rapidly to obtain the sum to the desired number of decimal places. For practical work the error may be estimated because of the asymptotic character of the series involved in the Euler-Maclaurin formula.

Write this in the form

$$\begin{aligned}(1)\quad \sum_{a}^{s-1} f(x) &= \int_a^s f(x)dx + \tfrac{1}{2}f(a) - \tfrac{1}{2}f(s) - \frac{f'(a)-f'(s)}{12} + \frac{f'''(a)-f'''(s)}{720} - \\ &\frac{f^{\mathrm{V}}(a)-f^{\mathrm{V}}(s)}{30240} + \frac{f^{\mathrm{VII}}(a)-f^{\mathrm{VII}}(s)}{1209600} - \cdots + (-1)^r B_r \frac{f^{(2r-1)}(a)-f^{(2r-1)}(s)}{(2r)!} + \cdots.\end{aligned}$$

If $s \to \infty$ one has accordingly in the ordinary case of convergence

$$(2)\quad \sum_{a}^{\infty} f(x) = \int_a^\infty f(x)dx + \tfrac{1}{2}f(a) - \frac{f'(a)}{12} + \frac{f'''(a)}{720} - \frac{f^{\mathrm{V}}(a)}{30240} + \cdots.$$

Now define $v(x) = \sum_{y=b}^{\infty} u(x, y) = \int_b^\infty u(x, y)\,dy + \tfrac{1}{2}u(x, b) - \frac{u_y(x, b)}{12} + \frac{u_{y^3}(x, b)}{720} - \cdots$ and $w(y) = \sum_{x=a}^{\infty} u(x, y) = \int_a^\infty u(x, y)dx + \tfrac{1}{2}u(a, y) - \frac{u_x(a, y)}{12} + \frac{u_{x^3}(a, y)}{720} - \cdots$, then $\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} u(x, y) = \sum_{x=1}^{a-1}\sum_{y=1}^{b-1} u(x, y) + \sum_{x=1}^{\infty} v(x) + \sum_{y=1}^{b-1} w(y)$

$$\begin{aligned}(3)\quad &= \int_1^\infty v(x)dx + \tfrac{1}{2}v(1) - \frac{v'(1)}{12} + \frac{v'''(1)}{720} - \frac{v^{\mathrm{V}}(1)}{30240} + \cdots + \sum_{x=1}^{a-1}\sum_{y=1}^{b-1} u(x, y) \\ &+ \int_1^b w(y)dy - \tfrac{1}{2}w(b) + \tfrac{1}{2}w(1) + \frac{w'(b)-w'(1)}{12} - \frac{w'''(b)-w'''(1)}{720} + \cdots.\end{aligned}$$

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[1] Presented to the Society, Nov. 30, 1934.

[2] W. F. Sheppard, "Some Quadrature Formulae," Proc. London Math. Soc., Vol. xxxii, 1900.

[3] J. O. Irwin, "Tracts for Computers," No. X, Cambridge Univ. Press, 1923, On Quadrature and Cubature.

[4] W. D. Baten, "A Remainder for the Euler-Maclaurin summation formula in two independent variables," Amer. Journal of Math., Vol. 54, 1932, pp. 265–275.