NOTE ON THE BINOMIAL DISTRIBUTION

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The purpose of this note is to show that

$$f(x) = (-1)^n \frac{q^n n!}{\pi} \left(\frac{p}{q}\right)^x \frac{\sin \pi x}{x^{(n+1)}} \tag{1}$$

where n is an integer $\geqq 0$, $0 < p < 1$, $p + q = 1$, and $x^{(n+1)} = x(x-1)(x-2) \cdots (x-n)$, is a function whose values at $x = 0, 1, 2, \cdots n$ are the successive terms of the expansion of $(q+p)^n$, and also to consider the problem of fitting $f(x)$ to an observed frequency distribution.

The statement made about (1) can be verified by evaluating (1) as an indeterminate form. On the other hand, (1) can be derived by observing that the x-th term (x an integer) of the expansion of $(q+p)^n$ is

$$\frac{n!}{x!(n-x)!} p^x q^{n-x} = \frac{\Gamma(n+1) p^x q^{n-x}}{\Gamma(x+1)\Gamma(n-x+1)}; \tag{2}$$

then (1) can be derived from (2) by means of the product expansions for $\Gamma(x)$ and $\sin x$. This derivation of (1) from (2) can also be carried out by expressing (2) as a Beta function and then using

$$B(x+1, n-x+1) = \int_0^\infty \frac{t^x}{(1+t)^{n+2}}\, dt = (-1)^n \frac{\pi}{(n+1)!} \frac{x^{(n+1)}}{\sin \pi x}.$$

This integration can be performed by means of the theory of residues.

Consider the problem of fitting (1) to an observed frequency distribution. We shall write (1) in the form

$$F(z) = ab^x \frac{\sin \pi x}{x^{(n+1)}}, \qquad x = \frac{nb}{1+b} + h(z - \bar{z}) \tag{3}$$

and determine the constants a, b, n, and h so that, when $\bar{z}$ is the mean of the observed distribution, $F(z)$ will fit the distribution.

The values of a, b, n, and h can be determined by the method of moments. Let ν_2, ν_3, and ν_4, denote the usual second, third, and fourth moments of the distribution, which are calculated in the usual way (as in W. P. Elderton, *Frequency-Curves and Correlation*) and not adjusted by any procedure such as Sheppard's adjustments. Also, use the usual notation $\beta_1 = \frac{\nu_3^2}{\nu_2^3}$ and $\beta_2 = \frac{\nu_4}{\nu_2^2}$.