ON CERTAIN DISTRIBUTIONS DERIVED FROM THE MULTINOMIAL DISTRIBUTION[1]

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1. **Introduction.** With the multinomial distribution as a background, there may be derived a number of distributions which are of interest in certain practical applications. Several of these distributions are here presented and the theory is illustrated by specific examples.

2. **Preliminary data.** In the discussion of the distributions to be considered there are needed certain factorial sums whose values are now to be derived. In the following discussion only positive integral values (including zero) are to be considered.

There is desired the value, in terms of N, n, r, of

$$f_r(n, N) = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!} \tag{2.1}$$

where the summation is for all values of $x_1, x_2, \cdots, x_n$ such that $x_1 + x_2 + \cdots + x_n = N$ and no x is equal to r.

Let us first consider the case for $r = 0$; i.e., we desire a value for the sum in (2.1) for all values of $x_1, x_2, \cdots, x_n$ such that $x_1 + x_2 + \cdots + x_n = N$ and no x is equal to zero. By the multinomial theorem, we have that[2]

$$(a_1 + a_2 + \cdots + a_n)^N = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!} a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n} \tag{2.2}$$

where the summation is for all values of $x_1, x_2, \cdots, x_n$ such that $x_1 + x_2 + \cdots + x_n = N$. If $a_1 = a_2 = \cdots = a_n = 1$, then

$$n^N = \sum \frac{N!}{x_1!\, x_2! \cdots x_n!}, \qquad x_1 + x_2 + \cdots + x_n = N. \tag{2.3}$$

The sum in (2.3) may however be rearranged into the sum of a number of terms as follows:

$$\tag{2.4} \begin{cases} \displaystyle\sum \frac{N!}{x_1!\, x_2! \cdots x_n!}, & x_1 + x_2 + \cdots + x_n = N, \quad \text{no } x = 0; \\ \displaystyle n \sum \frac{N!}{x_1!\, x_2! \cdots x_{n-1}!}, & x_1 + x_2 + \cdots + x_{n-1} = N, \quad \text{no } x = 0; \\ \displaystyle \frac{n(n-1)}{2} \sum \frac{N!}{x_1!\, x_2! \cdots x_{n-2}!}, & x_1' + x_2 + \cdots + x_{n-2} = N, \quad \text{no } x = 0; \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \displaystyle \binom{n}{r} \sum \frac{N!}{x_1!\, x_2! \cdots x_{n-r}!}, & x_1 + x_2 + \cdots + x_{n-r} = N, \quad \text{no } x = 0. \end{cases}$$

[1] Presented to the Institute of Mathematical Statistics January 2, 1936.

[2] H. S. Hall & S. R. Knight, *Higher Algebra*, MacMillan & Co., 4th Ed. (1924), Chap. 15.