ON THE POLYNOMIALS RELATED TO THE DIFFERENTIAL EQUATION $\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} \equiv \frac{N}{D}$

By Frank S. Beale

Introduction. In a previous issue of this Journal,[1] E. H. Hildebrandt has established the existence of a general system of polynomials $P_n(k, x)$ associated with the solutions of Pearson's Differential Equation

$$\text{(R)} \qquad \frac{1}{y}\frac{dy}{dx} = \frac{N}{D},$$

N and D being polynomials in x of degrees not exceeding one and two respectively with no factor in common.

It was shown that the polynomials $P_n(k, x) \equiv P_n$ themselves satisfy certain differential equations and a recurrence relation. The classical polynomials of Hermite, Legendre, Laguerre, and Jacobi are special types of $P_n(k, x)$. Since the classical polynomials are employed rather extensively in statistical theory, certain of their properties are of special interest.

It is the purpose of this paper to determine from Hildebrandt's general equations some new properties of $P_n(k, x)$ and to apply these properties to the classical polynomials. The paper consists of two parts. In part I some theorems are established concerning common zeros of D and P_n. In particular, a theorem is established to exhibit the conditions under which the zeros of P_n, which are not zeros of D, are simple. In part II a method is outlined for the classical polynomials by which one can determine the number and location of the real zeros in the various segments into which the zeros of D divide the x axis. The points of inflexion and the degree of the polynomials are also considered.

A new feature of the method employed is, we believe, its being based upon the use of differential equations of first order, for most part, while other investigators[2] have employed differential equations of second order. As to the results obtained, the author believes them to be partly new. They have points in common with the results of Fujiwara, Lawton and Webster.

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[1] Systems of Polynomials Connected with the Charlier Expansions, etc., Annals of Math. Stat., Vol. II, 1931, pp. 379–439.

[2] *M. Fujiwara:* On the zeros of Jacobi's Polynomials, Japanese Journal of Math., Vol. 2, 1925, pp. 1, 2.

W. Lawton: On the zeros of Certain Polynomials Related to Jacobi and Laguerre Polynomials, Bull. Am. Math. Soc., Vol. 38, 1932, pp. 442–449.

M. S. Webster: Thesis, Univ. of Penna. These results were kindly communicated to me by Dr. Webster.