ON DIFFERENTIAL OPERATORS DEVELOPED BY O'TOOLE

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1. O'Toole in his paper 'Symmetric Functions and Symmetric Functions of Symmetric Functions' [Ann. Statist. 2. (1931)102–49], has expressed Monomial Symmetric Functions $\sum a^{p_1} b^{p_2} c^{p_3} \cdots$, in terms of power-sums, s_r.

The Monomial Symmetric Functions can be written in partition notation as $\binom{k_1\, k_2\, k_3}{p_1 p_2 p_3} \cdots)$ where $k_1, k_2, \cdots$ denote the repetitions of parts.

To express $\binom{k_1\, k_2\, k_3}{p_1 p_2 p_3} \cdots)$ as a function of s_r', O'Toole has developed operators d_r and D_r, connected by the formulae,[1]

$$d_r = \frac{d}{ds_r},$$

$$\text{(A)} \qquad rd_r = \frac{(-1)^{r+1} \sum (-1)^{r+k}(k-1)!\, r \cdot D_A^{k_1} D_B^{k_2} \cdots}{k_1!\, k_2! \cdots};$$

$$\text{(B)} \qquad r!\, D_r = \frac{\sum r!\, d_A^{k_1} d_B^{k_2} \cdots}{k_1!\, k_2! \cdots},$$

$$\text{where } k_1 A + k_2 B + \cdots = r$$

$$k_1 + k_2 + \cdots = k.$$

In this paper it will be shown that these operational relations are easily deduced from the operators d_r and D_r of Hammond, used for expressing Monomial Symmetric Functions as functions of Elementary Symmetric Functions, a_r.

For the sake of distinction I shall use q_r and Q_r for the operators employed by O'Toole and keep d_r and D_r for Hammond's Operators.

Macmahon has dealt with Hammond's operators in his Combinatory Analysis Vol. I. Cambridge University Press (1915), where they are defined[2] as

$$D_r = \frac{1}{r!}(d_1^r) \qquad \text{and} \qquad d_r = \frac{d}{da_r} + a_1 \frac{d}{da_{r+1}} + a_2 \frac{d}{da_{r+2}} + \cdots, \qquad (1).$$

2. It is known[3] that

$$\log(1 - a_1 x + a_2 x^2 - a_3 x^3 + \cdots) = -\left(s_1 x + \frac{1}{2} s_2 x^2 + \cdots + \frac{1}{r} s_r x^r + \cdots\right).$$

[1] O'Toole, Loc. cit., p. 120.
[2] Macmahon. Comb. Analysis. I. 27–28.
[3] Ibid., p. 6.