ON THE INDEPENDENCE OF CERTAIN ESTIMATES OF VARIANCE[1]

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1. Introduction. It is well known that a necessary and sufficient condition that several statistics be independent in the probability sense, is that the characteristic function of the joint distribution of these statistics shall equal identically the product of the characteristic functions of the distributions of the individual statistics. Thus, if $x_1, x_2, \cdots, x_N$ are N independently observed values of a variable x which is subject to the distribution function $f(x)$, and if $\theta_1, \theta_2, \cdots, \theta_s$ are s statistics, each computed from the N observed values of x, the characteristic function of the joint distribution of the s statistics is given by

$$\varphi(t_1, t_2, \cdots, t_s) = \int \cdots \int e^{it_1\theta_1 + \cdots + it_s\theta_s} f(x_1) \cdots f(x_N)\, dx_N \cdots dx_1.$$

Here, $i = \sqrt{-1}$ and the limits of integration are taken so as to include all admissible values of x. Since the characteristic function of the distribution of θ_v, $v = 1, 2, \cdots, s$, is given by

$$\varphi_v(t_v) = \int \cdots \int e^{it_v\theta_v} f(x_1) \cdots f(x_N)\, dx_N \cdots dx_1,$$

the necessary and sufficient condition for the independence of the s statistics can be written

$$\varphi(t_1, \cdots, t_s) \doteq \varphi_1(t_1) \cdots \varphi_s(t_s), \tag{1}$$

for all real values of $t_1, t_2, \cdots t_s$.

An important phase of sampling theory in statistics is that in which the variable x is subject to the normal distribution function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \qquad -\infty \leq x \leq \infty,$$

and $\theta_1, \cdots, \theta_s$ are s real symmetric quadratic forms in the N independently observed values of x. That is,

$$\theta_1 = \sum_{j=1}^{N} \sum_{k=1}^{N} a_{jk} x_j x_k,$$

$$\theta_2 = \sum_{j=1}^{N} \sum_{k=1}^{N} b_{jk} x_j x_k,$$

$$\vdots$$

$$\theta_s = \sum_{j=1}^{N} \sum_{k=1}^{N} p_{jk} x_j x_k,$$

[1] Presented to the Institute of Mathematical Statistics on December 30, 1937, at the invitation of the progam committee. In the paper, we discuss, from a slightly different point of view, some of the material found in the references given at the close of the paper.