ON A CLASS OF DISTRIBUTIONS THAT APPROACH THE NORMAL DISTRIBUTION FUNCTION[1]

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1. **Formulation of the Problem.** An important property of a sequence of binomial coefficients is that, when suitably normalized and transformed, it converges to the normal distribution.[2] The object of this paper is to exhibit a large class of other sequences which also possess this property.

The Pascal recurrence formula may be taken as the defining property of the binomial coefficients. Let the combination of n things taken x at a time be denoted by $\binom{n}{x}$. If we set $f_n(x) = (\frac{1}{2})^n \cdot \binom{n}{x}$ for $0 \leq x \leq n$ and $f_n(x) = 0$ for $x < 0$ or $x > n$, then $f_n(x)$ is defined for all integers x. With this notation Pascal's recurrence formula, $\binom{n}{x} = \binom{n-1}{x} + \binom{n-1}{x-1}$, may be written

$$f_n(x) = \tfrac{1}{2}\,[f_{n-1}(x) + f_{n-1}(x-1)], \tag{1}$$

where this new form is valid for all integers x extending from $-\infty$ to $+\infty$.

In order to generalize, we may consider a sequence of distributions $f_1(x)$, $f_2(x), \cdots, f_n(x), \cdots$ each defined in terms of the preceding one by means of the recurrence formula

$$f_n(x) = \frac{1}{a_n + 1}\,[f_{n-1}(x-0) + f_{n-1}(x-1) + f_{n-1}(x-2) + \cdots + f_{n-1}(x-a_n)], \tag{2}$$

where the x are integers, and a_n is a positive integer which may change in value from one distribution to the next. The problem is to find conditions under which $f_n(x)$, in normalized form, approaches the normal distribution. The normalization of $f_n(x)$ is effected by the affine transformation

$$u = \frac{x - \bar{x}_n}{\sigma_n}; \qquad \varphi_n(u) = f_n(x), \tag{3}$$

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[2] Due to DeMoivre, 1731. By a variable distribution approaching the normal distribution, we mean that the integral under the variable distribution between any two limits approaches the corresponding integral under the normal curve.