THE ESTIMATION OF A QUOTIENT WHEN THE DENOMINATOR IS NORMALLY DISTRIBUTED

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1. Introduction. In an oceanographic investigation we have to deal with a time series consisting of single pairs of observed values x, y, of two independent stochastic variables, whose true (mean) values we shall denote respectively by a, b. Of interest is the corresponding time series of quotients (b/a), which it is required to estimate from the observations x, y. Both x and y are approximately normally distributed about their mean values a, b with rather large variances σ_x^2, σ_y^2 which can be estimated. It is easily possible for x to vanish or even to be of opposite sign to a, although a cannot itself vanish. The required estimates of (b/a) should have the property that they can be numerically integrated, i.e. that an arbitrary sum of such estimates shall equal the corresponding estimate of the true sum.

Let us define a function $\gamma(x)$ to have the property that its mathematical expectation $E\{\gamma(x)\}$ is exactly $1/a$, where $a = E(x)$. If such a function exists we shall have

$$E\{y\cdot\gamma(x)\} = E(y)\cdot E\{\gamma(x)\} = b\cdot(1/a) = b/a \tag{1}$$

so that $y\cdot\gamma(x)$ will be an estimate of b/a which has the required property: namely such estimates can be added, and we have

$$E\{y_1\gamma(x_1) + y_2\gamma(x_2)\} = E\{y_1\gamma(x_1)\} + E\{y_2\gamma(x_2)\} = b_1/a_1 + b_2/a_2$$

as required. It turns out that if x is normally distributed with non-zero mean such a function $\gamma(x)$ does exist, and is given by the formula

$$\gamma(x) = \frac{1}{\sigma_x}\exp(x^2/2\sigma_x^2)\int_{x/\sigma_x}^{\infty} e^{-t^2/2}\,dt = \frac{1}{\sigma_x}R_{x/\sigma_x} \tag{2}$$

where R_u is the "ratio of the area to the bounding ordinate" which is tabulated by J. P. Mills,[1] also in Pearson's tables.[2] Equation (2) holds if a is positive; if a is negative the integration should extend over $(x/\sigma_x, -\infty)$. It is easy to verify that

$$E(\gamma(x)) = \frac{1}{\sqrt{2\pi}\,\sigma_x}\int_{-\infty}^{\infty}\gamma(x)\exp\left(-\frac{(x-a)^2}{2\sigma_x^2}\right)dx = \frac{1}{a} \tag{3}$$

by direct substitution from (2).

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[1] J. P. Mills, "Table of ratio: area to bounding ordinate, for any portion of the normal curve," *Biometrika*, Vol. 18 (1926), pp. 395–400.

[2] Karl Pearson, *Tables for Statisticians and Biometricians*, part II, table III, Cambridge Univ. Press.