CONTINUED FRACTIONS FOR THE INCOMPLETE BETA FUNCTION[1]

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1. Introduction. Existing literature on the problem of calculating the incomplete Beta function

$$\mathrm{B}_x(p, q) = \int_0^x x^{p-1}(1-x)^{q-1}\,dx, \qquad 0 < x < 1, p > 0, q > 0, \tag{1.1}$$

and the levels of significance of Fisher's z [1] leave further work to be done. Müller's continued fraction and a new continued fraction are shown to possess complementary features covering the range of $\mathrm{B}_x(p, q)$ for all values of x, p, q. Previous methods of computing $I_x(p, q) = \mathrm{B}_x(p, q)/\mathrm{B}(p, q)$ are given in [2], [5], [6], [8], [10], [13], [14], [15].

Müller's continued fraction is

$$I_x(p, q) = C\left[\frac{b_1}{1+}\,\frac{b_2}{1+}\,\frac{b_3}{1+}\,\frac{b_4}{1+}\cdots\right], \tag{1.2}$$

where

$$C = \frac{\Gamma(p+q)}{\Gamma(p+1)\Gamma(q)}\,x^p(1-x)^{q-1}, \qquad b_1 = 1, \qquad \mu_s = \frac{q-s}{p+s},$$

$$b_{2s} = -\frac{(p+s-1)(p+s)}{(p+2s-2)(p+2s-1)}\,\mu_s\,\frac{x}{1-x},$$

$$b_{2s+1} = \frac{s(p+q+s)}{(p+2s-1)(p+2s)}\,\frac{x}{1-x}.$$

A convergent infinite series $1 + \sum_{n=1}^{\infty} d_n x^n$ can be converted into an infinite continued fraction of the form $\frac{1}{1+}\,\frac{c_1 x}{1+}\,\frac{c_2 x}{1+}\cdots$ where [4], [9] p. 304,

$$\begin{aligned} c_1 &= -\beta_1, & c_2 &= \frac{-\beta_2}{\beta_1}, \\ c_{2s} &= \frac{-\beta_{2s-3}\beta_{2s}}{\beta_{2s-2}\beta_{2s-1}}, & c_{2s+1} &= \frac{-\beta_{2s-2}\beta_{2s+1}}{\beta_{2s-1}\beta_{2s}}, \qquad s > 2 \end{aligned} \tag{1.3}$$

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