SOME GENERALIZATIONS OF THE LOGARITHMIC MEAN AND OF SIMILAR MEANS OF TWO VARIATES WHICH BECOME INDETERMINATE WHEN THE TWO VARIATES ARE EQUAL

By Edward L. Dodd

University of Texas

1. Introduction. The logarithmic mean m of positive numbers, x and y, as given by

$$(1) \qquad m = \frac{y - x}{\log_e y - \log_e x} = \frac{y - x}{\log_e (y/x)}$$

is of considerable importance in problems[1] relating to the flow of heat.

The logarithmic mean arises, moreover, in less technical problems such as the following: Given that incomes t in the interval, $x \leqq t \leqq y$, are distributed with frequency inversely proportional to t. That is, with $k = a$ positive constant,

$$(2) \qquad \phi(t)\, dt = (k/t)\, dt$$

is the number of individuals with incomes lying between t and $t + dt$. Then, with $x > 0$, the total number f of individual incomes is

$$(3) \qquad f = \int_x^y \phi(t)\, dt = k(\log y - \log x).$$

The combined income g of the group is

$$(4) \qquad g = \int_x^y t\phi(t)\, dt = k(y - x).$$

And thus the *logarithmic* mean g/f of the *two* numbers x and y in (1) is the *arithmetic mean of all* the incomes; that is, the *average income*—at least to a close approximation if the group is large enough that integration may replace summation.

Now m in (1) becomes *indeterminate*, if $x = y$. Nevertheless, if $c > 0$, and $x \to c$ and $y \to c$, then $m \to c$. Thus, we may properly speak of m as a mean of these two variates, x and y.

This logarithmic mean is one of a set of means studied by Renzo Cisbani[2], the general form being

[1] See Walker, Lewis, and McAdams, *Principles of Chemical Engineering*, McGraw Hill & Co., Part IV, Logarithmic mean temperature difference.

[2] R. Cisbani, "Contributi alla teoria delle medie." *Metron*, Vol. 13(1938), pp. 23–34.