A CHARACTERIZATION OF THE NORMAL DISTRIBUTION

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1. In sampling from a normal population the distributions of the mean and of the variance are mutually independent. This well known property of the normal distribution is used in deriving the distribution of "Student's" ratio. The independence of the distributions of the mean and of the variance characterizes the normal distribution. To show this one has to prove the following statement:

A necessary and sufficient condition for the normality of the parent distribution is that the sampling distributions of the mean and of the variance be independent.

That this condition is necessary follows from the above mentioned property of the normal distribution; so there is only to prove that this condition is sufficient. This was first proved by R. C. Geary[1] by using some of R. A. Fisher's general formulae for the seminvariants. However, a different proof, using characteristic functions might be of some interest.

2. Let $f(x)$ be the density function of a continuous probability distribution and let $x_1, x_2, \cdots, x_n$ be n observations of the variate x. Denote by $\bar{x} = \sum_{x=1}^{n} x_\alpha/n$ the sample mean, and by

$$s^2 = \sum_{\alpha=1}^{n} (x_\alpha - \bar{x})^2/n = [(n-1)\sum_{1}^{n} x_\alpha^2 - 2\sum_{\alpha=1}^{n-1}\sum_{\beta=\alpha}^{n-1} x_\alpha x_{\beta+1}]/n^2$$

the sample variance of these observations. The characteristic function of the distribution is then given by

$$\psi(t) = \int e^{itx} f(x)\, dx. \tag{1}$$

The characteristic function of the joint distribution of the statistics $\bar{x}$ and s^2 is known to be

$$\varphi(t_1, t_2) = \int \cdots \int e^{it_1\bar{x}+it_2s^2} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n. \tag{2}$$

In the same way one obtains the characteristic function of the mean $\bar{x}$ as

$$\varphi_1(t_1) = \varphi(t_1, 0) = \int \cdots \int e^{it_1\bar{x}} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n, \tag{2a}$$

[1] R. C. Geary, "Distribution of Student's ratio for nonnormal samples," *Roy. Stat. Soc. Jour.*, Supp. Vol. 3, no. 2.