THE CONSTRUCTION OF ORTHOGONAL LATIN SQUARES[1]

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A Latin square is an arrangement of m variables $x_1, x_2, \cdots, x_m$ into m rows and m columns such that no row and no column contains any of the variables twice. Two Latin squares are called orthogonal if when one is superimposed upon the other every ordered pair of variables occurs once in the resulting square.

The rows of a Latin square are permutations of the row $x_1, x_2, \cdots, x_m$. Let P_i be the permutation which transforms $x_1, x_2, \cdots, x_m$ into the ith row of the Latin square. Then $P_iP_j^{-1}$ leaves no variable unchanged for $i \neq j$. For otherwise one column would contain a variable twice. On the other hand each set of m permutations $P_1, P_2, \cdots, P_m$ such that $P_iP_j^{-1}$ leaves no variable unchanged generates a Latin square. We may therefore identify every Latin square with a set of m permutations $(P_1, P_2, \cdots, P_m)$ such that $P_iP_j^{-1}$ leaves no variable unchanged.

Now let $(P_1, P_2, \cdots, P_m)$, $(Q_1, Q_2, \cdots, Q_m)$ be a pair of orthogonal Latin squares. We shall show that $(P_1^{-1}Q_1, P_2^{-1}Q_2, \cdots, P_m^{-1}Q_m)$ is a Latin square. $P_i^{-1}Q_i$ is the transformation which transforms the ith row of $(P_1, P_2, \cdots, P_m)$ into the ith row of $(Q_1, Q_2, \cdots, Q_m)$. Since every pair of variables occurs exactly once if the second square is imposed upon the first, the square $(P_1^{-1}Q_1, P_2^{-1}Q_2, \cdots, P_m^{-1}Q_m)$ contains for every i and k a permutation which transforms x_i into x_k. But then it can not contain two permutations which transform x_i into x_k. This argument can be reversed and it follows that $(P_1, P_2, \cdots, P_m)$ and $(Q_1, Q_2, \cdots, Q_m)$ are orthogonal if and only if $(P_1^{-1}Q_1, P_2^{-1}Q_2, \cdots, P_m^{-1}Q_m)$ is a Latin square.

Denote now by an m sided square S any set of m permutations $(S_1, S_2, \cdots, S_m)$ and by the product SS' of two squares S and S' the square $(S_1S_1', S_2S_1', \cdots, S_mS_m')$. Then we can state: Two Latin squares L_1 and L_2 are orthogonal if and only if there exists a Latin square L_{12} such that

$$L_1L_{12} = L_2. \tag{1}$$

Now let $L_1, L_2, \cdots, L_r$ be a set of r mutually orthogonal Latin squares. Then we must have $L_iL_{ik} = L_k$ where L_{ik} is a Latin square if $i \neq k$. Hence we have the theorem

Theorem 1: *The Latin squares $L_1, L_2, \cdots, L_r$ are orthogonal if and only if there exist $r(r-1)$ Latin squares $L_{ik}(i \neq k)$ such that $L_iL_{ik} = L_k$.*

Corollary: *If L^i, L^k and L^{i-k} are Latin squares then L^i is orthogonal to L^k.*

For instance if L and L^2 are Latin squares then L is orthogonal to L^2.

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