GENERALIZED POISSON DISTRIBUTION

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1. Introduction. The Poisson distribution is one of the most fundamental of statistical distributions. It is the distribution law for the number of events if the probability of an event happening in any infinitesimal unit of time is independent of the probability of its happening in any other unit of time. Frequently when we analyze statistics which obey the Poisson law it is desirable to give varying weights to the different events instead of considering them all of equal value. Such is the case in analyzing insurance statistics where the events are the claims received by the office and the weights are the cost of the claim to the company. We shall now show how the Poisson distribution can be generalized so as to be adequate for such an analysis.

2. First development. Let $f(x, \alpha)$ be the distribution function of the weights assigned to the events where the variable, x, refers to the weight and the variable, α, refers to time. The characteristic function of $f(x, \alpha)$ is

$$\phi(t, \alpha) = \int e^{itx} f(x, \alpha)\, dx.$$

Also let $p(\alpha)\, d\alpha$ be the probability that an event will occur in the infinitesimal unit of time, α to $\alpha + d\alpha$. If y represents the sum of the weights, the distribution function of y for this unit of time is

$$\begin{aligned} F_{d\alpha}(y, \alpha) &= 1 - p(\alpha)\, d\alpha, \qquad & y = 0 \\ &= f(y, \alpha) p(\alpha)\, d\alpha, \qquad & y > 0. \end{aligned} \tag{1}$$

The characteristic function of this distribution is

$$\begin{aligned} \Phi_{d\alpha}(t, \alpha) &= e^{ito}(1 - p(\alpha)\, d\alpha) + p(\alpha)\, d\alpha \int e^{ity} f(y, \alpha)\, dy \\ &= 1 - p(\alpha)\, d\alpha(1 - \phi(t, \alpha)) \\ &= e^{-p(\alpha)d\alpha(1-\phi(t,\alpha))}. \end{aligned} \tag{2}$$

In forming equations (1) and (2) we ignore infinitesimals of orders higher than the first in the $d\alpha$.

The expected number of events in the period of time from α_1 to α_2 is

$$P = \int_{\alpha_1}^{\alpha_2} p(\alpha)\, d\alpha,$$

and the mean distribution of weights during the same period of time is

$$f(x) = \int_{\alpha_1}^{\alpha_2} [p(\alpha)/P] f(x, \alpha)\, d\alpha.$$