GENERALIZATION OF POINCARÉ'S FORMULA IN THE THEORY OF PROBABILITY

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Let $p_{[m]}(1, \cdots, n)$, $(0 \leq m \leq n)$ denote the probability of the occurrence of exactly m events among the n arbitrary events $E_1, \cdots, E_n$; and $p_m(1, \cdots, n)$ $(1 \leq m \leq n)$ that of at least m. Let $p_{\nu_1 \cdots \nu_i}$ $(1 \leq i \leq n)$, where $(\nu_1 \cdots \nu_i)$ is a combination (without repetition) out of $(1, \cdots, n)$, denote the probability of the occurrence of $E_{\nu_1}, \cdots, E_{\nu_i}$ (without regard to the other events); and

$$S_0 = 1, \qquad S_i = \sum_{(\nu_1 \cdots \nu_i)} p_{\nu_1 \cdots \nu_i},$$

where the summation extends to all the combinations with i members out of $(1, \cdots, n)$.

Then Poincaré's formula may be written as follows:

$$p_{[0]}(1, \cdots, n) = \sum_{i=0}^{n} (-1)^i S_i .$$

An equivalent formula is:

$$p_1(1, \cdots, n) = \sum_{i=1}^{n} (-1)^{i-1} S_i .$$

The following conventions concerning the binomial coefficients are made:

$$\binom{0}{0} = 1, \qquad \binom{a}{b} = 0 \qquad \text{if } a < b \text{ or } b < 0.$$

Two generalizations, possibly due to de Mises, are

$$p_{[m]}(1, \cdots, n) = \sum_{i=m}^{n} (-1)^{(i-m)} \binom{i}{m} S_i ;$$

$$p_m(1, \cdots, n) = \sum_{i=m}^{n} (-1)^{(i-m)} \binom{i-1}{m-1} S_i .$$

We notice that the probabilities appearing on the left-hand sides of these formulas are symmetrical with respect to the set of suffixes $(1, \cdots, n)$, and the sums on the right-hand sides are symmetrical in the same way.

As a natural generalization let us consider a probability which is symmetrical with respect to certain sub-sets of $(1, \cdots, n)$. We divide the n events into r sets:

$$E_{\nu_{11}}, \cdots, E_{\nu_{1n_1}}; E_{\nu_{21}}, \cdots, E_{\nu_{2n_2}}; \cdots ; E_{\nu_{r1}}, \cdots, E_{\nu_{rn_r}};$$

where $n_1 + n_2 + \cdots + n_r = n$. And we ask for the probability that out of the first set of n_1 events exactly m_1 events occur; and out of the second set of n_2 events exactly m_2 events occur; and so on; and finally, out of the rth set of n_r events exactly m_r events occur. When this problem is solved the analogous problem