ON STOCHASTIC LIMIT AND ORDER RELATIONSHIPS

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1. Introduction. The concept of a stochastic limit is frequently used in statistical literature. Writers of papers on problems in statistics and probability usually prove only those special cases of more general theorems which are necessary for the solution of their particular problems. Thus readers of statistical papers are confronted with the necessity of laboriously ploughing through details, a task which is made more difficult by the fact that no uniform notation has as yet been introduced. It is therefore the purpose of the present paper to outline a systematic theory of stochastic limit and order relationships and at the same time to propose a convenient notation analogous to the notation of ordinary limit and order relationships. The theorems derived in this paper are of a more general nature and seem to contain to the authors' knowledge all previous results in the literature. For instance the so-called δ-method for the derivation of asymptotic standard deviations and limit distributions, also two lemmas by J. L. Doob [1] on products, sums and quotients of random variables and a theorem derived by W. G. Madow [2] are special cases of our results. It is hoped that such a general theory together with a convenient notation will considerably facilitate the derivation of theorems concerning stochastic limits and limit distributions. In section 2 we define the notion of convergence in probability and that of stochastic order and derive 5 theorems of a very general nature. Section 2 contains 2 corollaries of these general theorems which have so far been most important in applications.

We shall frequently need the concept of a vector. A vector $a = (a^1, \cdots, a^r)$ is an ordered set of r numbers $a^1, \cdots, a^r$. The numbers $a^1, \cdots, a^r$ are called the components of a. If the components are random variables then the vector is called a random vector.

We shall generally denote by a, b constant vectors by x, y random vectors and by $a^1, \cdots, a^r, x^1, \cdots, x^r$ their components. Differing from the usual practice we shall put $|a| = (|a^1|, \cdots, |a^r|)$ and we shall write $a < b$ or $a \leq b$ if $a^i < b^i$ or $a^i \leq b^i$ for every i. This notation saves a great amount of writing, since all our theorems except theorem 4 are valid for sequences of any number of jointly distributed variates.

We shall review here the ordinary order notation. In all that follows let $f(N)$ be a positive function defined for all positive integers N.

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