ON THE MEASURE OF A RANDOM SET

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1. Introduction. The following is perhaps the simplest non-trivial example of the type of problem to be considered in this paper. On the real number axis let N points x_i $(i = 1, 2, \cdots, N)$ be chosen independently and by the same random process, so that the probability that x_i shall lie to the left of any point x is a given function of x,

$$\sigma(x) = \Pr(x_i < x). \tag{1}$$

With the points x_i as centers, N unit intervals are drawn. Let X denote the set-theoretical sum of the N intervals, and let $\mu(X)$ denote the linear measure of X. Then $\mu(X)$ will be a chance variable whose values may range from 1 to N, and whose probability distribution is completely determined by $\sigma(x)$. Let $\tau(u)$ denote the probability that $\mu(X)$ be less than u. Then by definition, the expected value of $\mu(X)$ is

$$E(\mu(X)) = \int_1^N u \, d\tau(u), \tag{2}$$

where

$$\tau(u) = \Pr(\mu(X) < u). \tag{3}$$

The problem is to transform the expression for $E(\mu(X))$ so that its value may be computed in terms of the given function $\sigma(x)$.

In order to do this, we observe that, since the x_i are independent,

$$\tau(u) = \int \cdots \int_{C(u)} d\sigma(x_1) \cdots d\sigma(x_N), \tag{4}$$

where the domain of integration $C(u)$ consists of all points $(x_1, \cdots, x_N)$ in Euclidean N-dimensional space such that the linear measure of the set-theoretical sum of N unit intervals with centers at the points x_i is less than u. Here, however, a difficulty arises. Due to the possible overlapping of the intervals, the geometrical description of the domain $C(u)$ is such as to make the explicit evaluation of the integral (4) a complicated matter.

The difficulty is even more serious in the analogous problem where instead of N unit intervals on the line we have N unit circles in the plane, with a given probability distribution for their centers (x_i, y_i). Again we seek the expected value of the measure of the set-theoretical sum of the N circles. The corresponding domain $C(u)$ in $2N$-dimensional space will now be very complicated.

It is the object of this paper to show how, in such cases as these, the expected value of $\mu(X)$ may be found without first finding the distribution function $\tau(u)$.