ASYMPTOTIC DISTRIBUTION OF RUNS UP AND DOWN[1]

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1. Introduction. Let $a_1, a_2, \cdots, a_n$ be any n unequal numbers and let $S = (h_1, h_2, \cdots, h_n)$ be a random permutation of them, with each permutation having the same probability, which is therefore $\frac{1}{n!}$. Let R be the sequence of signs ($+$ or $-$) of the differences $h_{i+1} - h_i$ ($i = 1, 2, \cdots, n - 1$). Then R is also a chance variable. A sequence of p successive $+$ ($-$) signs not immediately preceded or followed by a $+$ ($-$) sign is called a run up (down) of length p. The term "run" applies to both runs up and runs down. As an example, if $S = (4\ 6\ 2\ 3\ 5)$, then in $R = (+\ -\ +\ +)$ there are three runs, one up of length one, one down of length one, and one up of length two.

The purpose of this paper is to establish several theorems about the limiting distributions of a class of functions of runs up and down. These results are applicable to certain techniques which have been employed in quality control and the analysis of economic time series. They are also shown to apply to a large class of "runs."

2. Joint distribution of runs of several lengths. Let r_p be the number of runs of length p in R and r'_p the number of runs of length p or more in R. Then r_p and r'_p are chance variables. The expectations $E(r_p)$ and $E(r'_p)$, the variances $\sigma^2(r_p)$ and $\sigma^2(r'_p)$, and the covariances $\sigma(r_{p_1}r_{p_2})$ are given by Levene and Wolfowitz [1]. They are all of the order n. Let

$$y_p = \frac{r_p - E(r_p)}{\sqrt{n}},$$

$$y'_p = \frac{r'_p - E(r'_p)}{\sqrt{n}}.$$

Our first results are embodied in the following theorem:

THEOREM 1. *Let l be any non-negative integer. The joint distribution of $y_1, \cdots, y_l, y'_{(l+1)}$, approaches the normal distribution as $n \to \infty$.*

We shall give the proof for the case $l = 1$, but it will easily be seen to be perfectly general.

Let $x_{pi} = 1$ if the sign ($+$ or $-$) of $h_{i+1} - h_i$ is the initial sign of a run of length p, and let $x_{pi} = 0$ otherwise. Let $w_{pi} = 1$ if the sign of $h_{i+1} - h_i$ is the

[1] Part of the results of this paper was presented to the Institute of Mathematical Statistics and the American Mathematical Society at their joint meeting in New Brunswick, N. J., on September 13, 1943.