NOTE ON A LEMMA

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In a previous paper on the power function of the analysis of variance test[1], the author stated the following lemma (designated there as Lemma 2):

Lemma 2. *Let $v_1, \cdots, v_k$ be k normally and independently distributed variates with a common variance σ^2. Denote the mean value of v_i by α_i $(i = 1, \cdots, k)$ and let $f(v_1, \cdots, v_k, \sigma)$ be a function the variables $v_1, \cdots, v_k$ and σ which does not involve the mean values $\alpha_1, \cdots, \alpha_k$. Then, if the expected value of $f(v_1, \cdots, v_k, \sigma)$ is equal to zero, $f(v_1, \cdots, v_k, \sigma)$ is identically equal to zero, except perhaps on a set of measure zero.*

In the paper mentioned above it was intended to state this lemma for bounded functions $f(v_1, \cdots, v_k)$ and the lemma was used there only in a case where $f(v_1, \cdots, v_k)$ is bounded. Through an oversight this restriction on $f(v_1, \cdots, v_k)$ was not stated explicitly.[2] The published proof of Lemma 2 is adequate if $f(v_1, \cdots, v_k)$ is assumed to be bounded. From the fact that the moments of a multivariate normal distribution determine uniquely the distribution it is concluded there that if for any set $(r_1, \cdots, r_k)$ of non-negative integers

$$(1) \qquad \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} v_1^{r_1} \cdots v_k^{r_k} f(v_1, \cdots, v_k) e^{-\frac{1}{2}\Sigma(v_i-\alpha_i)^2}\, dv_1 \cdots dv_k = 0$$

identically in the parameters $\alpha_1, \cdots, \alpha_k$ then $f(v_1, \cdots, v_k)$ must be equal to zero except perhaps on a set of measure zero. This conclusion is obvious if $f(v_1, \cdots, v_k)$ is bounded. In fact, from (1) and the boundedness of $f(v_1, \cdots, v_k)$ it follows that there exists a finite value A such that

$$\varphi(v_1, \cdots, v_k) = \frac{1}{(2\pi)^{k/2}}\left[1 - \frac{1}{A} f(v_1, \cdots, v_k)\right] e^{-\frac{1}{2}\Sigma(v_i-\alpha_i)^2}$$

is a probability density function with moments equal to those of the normal distribution

$$\psi(v_1, \cdots, v_k) = \frac{1}{(2\pi)^{k/2}} e^{-\frac{1}{2}\Sigma(v_i-\alpha_i)^2}.$$

Hence $f(v_1, \cdots, v_k)$ must be equal to zero except perhaps on a set of measure zero. However, this conclusion is not so immediate if no restriction is imposed on $f(v_1, \cdots, v_k)$ except that

$$(2) \qquad \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |f(v_1, \cdots, v_k)|\, e^{-\frac{1}{2}\Sigma(v_i-\alpha_i)^2}\, dv_1 \cdots dv_k < \infty$$

for all values of the parameters $\alpha_1, \cdots, \alpha_k$. It is the purpose of this note to prove this. In other words, we shall prove the following proposition:

[1] A. Wald, "On the power function of the analysis of variance test," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 434.

[2] I wish to thank Prof. J. Neyman for calling my attention to this omission.