THE ELEMENTARY GAUSSIAN PROCESSES

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1. Introduction

One of the simplest interesting classes of temporally homogeneous stochastic processes is that for which the distributions of the defining chance variables $\{x(t)\}$ are Gaussian. It is supposed that

(A) if $t_1 < \cdots < t_\nu$, the multivariate distribution of $x(t_1), \cdots, x(t_\nu)$ is Gaussian,[1] and that

(B) this distribution is unchanged by translations of the t-axis.

The process is N-dimensional if $x(t)$ is an N-tuple $x_1(t), \cdots, x_N(t)$. The means $E\{x(t)\}$[2] are independent of t, and will always be supposed to vanish in the following discussion.

The correlation matrix function $R(t)$: $(r_{ij}(t))$ is defined by

$$r_{ij}(t) = E\{x_i(s)x_j(s+t)\}. \tag{1.1.1}$$

This expectation is independent of s, because of condition (B). The matrix function $R(t)$ satisfies the equation

$$r_{ij}(t) = r_{ji}(-t), \qquad i, j = 1, \cdots, N. \tag{1.1.2}$$

It follows that when $t = 0$ the matrix is symmetric:

$$r_{ij}(0) = r_{ji}(0), \qquad i, j = 1, \cdots, N, \tag{1.1.3}$$

and it is also well known that $R(0)$ is non-negative definite. Conditions on the functions $r_{ij}(t)$ necessary and sufficient that $R(t)$ be the correlation matrix function of a stochastic process were found for the case $N = 1$ by Khintchine[3] and for all N by Cramér.[4]

Hypothesis (A), that the process is Gaussian seems at first a restriction so strong that Gaussian processes are unimportant. These processes are, however, of fundamental importance, for the following reasons.

(i) If $R(t)$ is the correlation matrix function of *any* temporally homogeneous stochastic process, there is, according to Khintchine and Cramér, a Gaussian process with this same correlation function. This Gaussian process is uniquely determined by the correlation function (assuming that all first order moments vanish, as usual). Because of this intimate connection between the temporally homogeneous Gaussian processes and the most general temporally homogeneous

[1] Singular Gaussian distributions will not be excluded. For example the $x(t_j)$ may all vanish identically.

[2] The expectation of a chance variable x will be denoted by $E\{x\}$.

[3] *Matematische Annalen*, Vol. 109 (1934), p. 608.

[4] *Annals of Math.*, Vol. 41 (1940), pp. 215–230.