A NOTE ON THE BEHRENS-FISHER PROBLEM

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A commonly occurring problem of statistical inference is the comparison of the means of two normal universes when the ratio of their variances is unknown. Let $(x_1, \cdots, x_m)$ be a random sample from one normal population with mean α and variance μ, and $(y_1, \cdots, y_n)$, a random sample from another with mean β and variance ν. The problem is then that of making statistical inferences about the difference δ of the means, $\delta = \alpha - \beta$, when the ratio μ/ν is unknown. Convenient tests and confidence intervals are available if one can find a linear form L and a quadratic form Q in the vector $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ with coefficients independent of the unknown parameters α, β, μ, ν, such that for some positive integer k, the quotient

$$(L - \delta)/(Q/k)^{\frac{1}{2}} \tag{1}$$

has the t-distribution with k degrees of freedom. For this it is sufficient that the following conditions be satisfied for *all* values of the parameters: (*i*) *L and Q are independently distributed,* (*ii*)[1] $E(L) = \delta$, (*iii*) *Q/σ^2 has the χ^2-distribution with k degrees of freedom, where σ^2 is the variance of L.*

In a recent paper [1] the author investigated the Behrens-Fisher problem as delimited by the above three conditions,[2] and among other results arrived at a simple solution. This solution however does not have the property that the quotient (1) is symmetric, whereby in this note we shall mean the following: A function of the samples and parameters will be called symmetric if it is invariant under permutations of the x's among themselves and of the y's among themselves. Let us therefore formulate condition (*iv*): *the quotient* (1) *is symmetric.* Since (*iv*) would be extremely desirable, both for practical and theoretical reasons, and since the author has received several inquiries on this matter, it is considered worth while to outline a proof that conditions (*ii*) and (*iii*) imply that (*iv*) cannot be satisfied, in other words, there exists no "symmetric solution" of the Behrens-Fisher problem within the framework we have imposed. Perhaps this is a simple example of a larger class of problems in which an approach, natural in the light of past developments, forces us to an asymmetric solution.

Suppose (*iv*) is satisfied. By substituting special values for the vector $(x_1, \cdots, x_m, y_1, \cdots, y_n)$ and then making permutations allowed by (*iv*) we find that L and Q must be of the form

$$L = c_1 \sum_i x_i + c_2 \sum_j y_j, \tag{2}$$

$$Q = c_3 \sum_i x_i^2 + c_4 \sum_{i \neq i'} x_i x_{i'} + c_5 \sum_j y_j^2 + c_6 \sum_{j \neq j'} y_j y_{j'} + c_7 \sum_{i,j} x_i y_j, \tag{3}$$

[1] $E(f)$ denotes the expected value of f.

[2] Although these conditions appear simpler than those in [1] they may be shown equivalent.