MULTIPLE MATCHING AND RUNS BY THE SYMBOLIC METHOD

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1. Introduction. The two subjects in the title have generally been treated by distinct methods, an excellent summary of which is given by S. S. Wilks in Chapter X of [13]. For two-deck matching, an appreciable simplification over the classical work of MacMahon [7], which seems to underlie the generating function used by Wilks [12] and Battin [2], has been shown by one of us [5] to follow from symbolic methods. Here we give an elaboration of these methods to multiple matching and to runs.

The basis of the symbolic method in both problems has been given in [6], but for completeness a skeleton resume is given in Section 2 below. A new point is stressed: the relation of coefficients in polynomials of the symbolic method to factorial moments (cf. Fréchet [4]).

The emphasis for the most part is on showing the expedition of the symbolic method in reaching known results, but in several instances new results are obtained.

2. Symbolic expressions and moments. Let $A_1, \cdots, A_n$ be arbitrary events and let $p(A_{i_1}, \cdots, A_{i_k})$ denote the joint probability of $A_{i_1}, \cdots, A_{i_k}$; let P_r be the probability that exactly r of the events occur. Then

$$P_r = \sum_{k=0}^{n} (-1)^r {}_kC_r \Sigma(-1)^k p(A_{i_1}, \cdots, A_{i_k}) \tag{1}$$

and in particular

$$P_0 = \sum_{k=0}^{n} \Sigma(-1)^k p(A_{i_1}, \cdots, A_{i_k}),$$

or symbolically

$$P_0 = [1 - p(A_1)][1 - p(A_2)] \cdots [1 - p(A_n)]. \tag{2}$$

The cases to be studied will be exclusively ones where so-called *quasi-symmetry* holds, i.e., $p(A_{i_1}, \cdots, A_{i_k})$ is either 0 or a function ϕ_k of k alone. In that event (2) can be evaluated as follows: suppress all products that vanish, and form a polynomial $f(E)$ by replacing each surviving term $p(A_i)$ by E. Then $P_0 = f(E)\phi_0$ where E is a displacement operator: $E^k\phi_0 = \phi_k$.

The same polynomial $f(E)$ can also be used to obtain P_r and the moments of the distribution. From (1) we see that $P_r = f(E)\psi_0$, where $\psi_k = (-1)^r {}_kC_r\phi_k$. Again it is well known (Fréchet [4]) that the k-th factorial moment, defined by

$$M_{(k)} = \sum_{i=0}^{n} i(i-1) \cdots (i-k+1)P_i,$$