A GENERALIZATION OF TSHEBYSHEV'S INEQUALITY TO TWO DIMENSIONS

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1. Let $X_1, X_2, \cdots, X_n$ be independent random variables with expectations $E(X_j) = e_j$ and variances $\sigma^2(X_j) = \sigma_j^2$ for $j = 1, 2, \cdots, n$. The question may be asked: What is the upper bound for the probability $P\left(\sum_{j=1}^{n} \frac{(X_j - e_j)^2}{t_j^2} \geq 1\right)$ that the point $(X_1, X_2, \cdots, X_n)$ does not fall inside of the ellipsoid

$$\sum_{j=1}^{n} \frac{(X_j - e_j)^2}{t_j^2} = 1?$$

For $n = 1$ the answer to this question is given by Tshebyshev's inequality

$$P\left[\frac{(X - E(x))^2}{t^2} \geq 1\right] \leq \frac{\sigma^2(X)}{t^2} \tag{1.1}$$

which can not be improved without further assumptions. By a trivial generalization of the argument leading to (1.1) one can prove the inequality

$$P\left(\sum_{j=1}^{n} \frac{(X_j - e_j)^2}{t_j^2} \geq 1\right) \leq \sum_{j=1}^{n} \frac{\sigma_j^2}{t_j^2} \tag{1.2}$$

for any integer n. This inequality, however, can be improved for $n \geq 2$. In particular, for $n = 2$, the following theorem will be proved:

Theorem 1.1. *Let X and Y be independent random variables, with expectations $E(X) = X_0$, $E(Y) = Y_0$ and variances σ_X^2, σ_Y^2. Then, for any $s > 0$, $t > 0$ such that $\frac{\sigma_X^2}{s^2} \leq \frac{\sigma_Y^2}{t^2}$ we have*

$$P\left[\frac{(X - X_0)^2}{s^2} + \frac{(Y - Y_0)^2}{t^2} \geq 1\right] \leq L(s, t) \tag{1.3}$$

where

$$L(s, t) = \begin{cases} 1 \quad \text{if } \dfrac{\sigma_X^2}{s^2} + \dfrac{\sigma_Y^2}{t^2} \geq 1 \\[2ex] \dfrac{\sigma_X^2}{s^2} + \dfrac{\sigma_Y^2}{t^2} - \dfrac{\sigma_X^2}{s^2} \cdot \dfrac{1 - \left(\dfrac{\sigma_X^2}{s^2} + \dfrac{\sigma_Y^2}{t^2}\right)}{1 - \dfrac{\sigma_X^2}{s^2}} \quad \textit{if} \\[2ex] \qquad \dfrac{\sigma_X^2}{s^2} + \dfrac{\sigma_Y^2}{t^2} \leq 1 \leq \frac{1}{2}\left(\dfrac{\sigma_X^2}{s^2} + \dfrac{2\sigma_Y^2}{t^2} + \sqrt{\dfrac{\sigma_X^4}{s^4} + \dfrac{4\sigma_Y^4}{t^4}}\right) \\[2ex] \dfrac{\sigma_X^2}{s^2} + \dfrac{\sigma_Y^2}{t^2} - \dfrac{\sigma_X^2 \sigma_Y^2}{s^2 t^2} \quad \textit{if } \frac{1}{2}\left(\dfrac{\sigma_X^2}{s^2} + \dfrac{2\sigma_Y^2}{t^2} + \sqrt{\dfrac{\sigma_X^4}{s^4} + \dfrac{4\sigma_Y^4}{t^4}}\right) \leq 1. \end{cases} \tag{1.4}$$