A LOWER BOUND FOR THE VARIANCE OF SOME UNBIASED SEQUENTIAL ESTIMATES

By D. Blackwell and M. A. Girshick

Howard University and *Bureau of the Census*

Consider a sequence of independent chance variables $x_1, x_2, \cdots$ with identical distributions determined by an unknown parameter θ. We assume that $E\,x_i = \theta$ and that $W_k = x_1 + \cdots + x_k$ is a sufficient statistic for estimating θ from $x_1, \cdots, x_k$. A sequential sampling procedure is defined by a sequence of mutually exclusive events S_k such that S_k depends only on $x_1, \cdots, x_k$ and $\Sigma\, P(S_k) = 1$. Define $W = W_k$ and $n = k$ when S_k occurs. In a previous paper by one of the authors [1] it was shown that if $S_k = W_k\, C(S_1 + \cdots + S_{k-1})$, (where $C(A)$ denotes the event that A does not occur), the function $V(W, n) = E(x_1 \mid W, n)$ is an unbiased estimate of θ, and $\sigma^2(V) \leq \sigma^2(x_1)$. It is the purpose of this note to obtain a lower bound for $\sigma^2(V)$. Our result is:

Theorem I. $\sigma^2(V) \geq \dfrac{\sigma^2(x_1)}{E(n)}$.

We remark that the lower bound is actually attained in the classical case of samples of constant size N. For in this case, (see [1]), $V = E(x_1 \mid W_N) = W_N/N$. In fact we shall show that in a sense this is the only case in which the lower bound is attained.

The proof of Theorem I depends on certain properties of sums of independent chance variables. These, formulated more generally than is required for the proof of Theorem I, are given in

Theorem II. *Let $x_1, x_2, \cdots$ be independent chance variables with identical distributions, having mean θ and variance $\sigma^2(x_1)$. Let furthermore $\{S_k\}$ be any sequential test for which $E(n)$ is finite. Let $W = x_1 + \cdots + x_k$ when $n = k$. Then*

(a) $\sigma^2(W - \theta n) \leq \sigma^2(x_1)\, E(n)$.

(b) *If $\sigma^2(n)$ is finite, the equality sign holds in* (a).

(c) $E[x_1(W - \theta n)] = \sigma^2(x_1)$.

Proof of (a). Write $y_i = x_i - \theta$, and define $Y = y_1 + \cdots + y_k$ when $n = k$. By definition,

$$\sigma^2(W - \theta n) = \sum_{k=1}^{\infty} \int_{S_k} (y_1 + \cdots + y_k)^2\, dP. \tag{1}$$

To prove (a), we must verify that the series on the right of expression (1) converges and has sum $\leq \sigma^2(x_1)E(n)$. Now

$$\begin{aligned} &\sum_{k=1}^{N} \int_{S_k} (y_1 + \cdots + y_k)^2\, dP \\ &\leq \sum_{k=1}^{N-1} \int_{S_k} (y_1 + \cdots + y_k)^2\, dP + \int_{n \geq N} (y_1 + \cdots + y_N)^2\, dP \\ &= \sum_{k=1}^{N} \int_{n \geq k} y_k^2\, dP + 2 \sum_{k=2}^{N} \int_{n \geq k} y_k(y_1 + \cdots + y_{k-1})\, dP. \end{aligned} \tag{2}$$