

ON THE USE OF THE NON-CENTRAL t -DISTRIBUTION FOR COMPARING PERCENTAGE POINTS OF NORMAL POPULATIONS

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1. Introduction. Consider two normal populations with the same variance and means μ and ν respectively. It is well known that confidence intervals and significance tests can be obtained for the difference $\mu - \nu$. Since μ is the 50% point of the first population and ν is the 50% point of the second population, this represents a particular solution of the general problem of obtaining confidence intervals and significance tests for the difference $\theta_\alpha - \varphi_\beta$, where θ_α is the α percent point of the first population and φ_β is the β percent point of the second population. The purpose of this note is to point out that the results of Johnson and Welch [1] for the non-central t -distribution can be used to furnish a solution of the general problem.

2. Analysis. Let A_γ be the γ percent point of the normal population with zero mean and unit variance (i.e. exactly $\gamma\%$ of the population has values less than A_γ). Then if σ is the common standard deviation,

$$\theta_\alpha = \mu + A_\alpha \sigma, \quad \varphi_\beta = \nu + A_\beta \sigma.$$

Thus

$$\theta_\alpha - \varphi_\beta = (\mu - \nu) + (A_\alpha - A_\beta) \sigma.$$

The non-central t -distribution investigated by Johnson and Welch in [1] is based on the quantity

$$t = (z + \delta) / \sqrt{\chi^2/f},$$

where z has a normal distribution with zero mean and unit variance, δ is a constant, and χ^2 has a χ^2 -distribution with f degrees of freedom and is distributed independently of z . Methods and tables are given in [1] whereby a constant $t(f, \delta, \epsilon)$ can be computed having the property that

$$Pr[t > t(f, \delta, \epsilon)] = \epsilon.$$

These relations will be used to obtain confidence intervals for $\theta_\alpha - \varphi_\beta$. The resulting confidence intervals can be used to obtain significance tests for $\theta_\alpha - \varphi_\beta$.

Let $x_1, \dots, x_n$ be a random sample of size n from the first population while $y_1, \dots, y_m$ is a random sample of size m from the second population. Then consider

$$\begin{aligned} & \frac{\bar{x} - \bar{y} - (\theta_\alpha - \varphi_\beta)}{\sqrt{\sum_1^n (x_i - \bar{x})^2 + \sum_1^m (y_j - \bar{y})^2}} \cdot \sqrt{\frac{m+n-2}{\frac{1}{n} + \frac{1}{m}}} \\ &= \frac{\left[\frac{\bar{x} - \bar{y} - (\mu - \nu)}{\sigma \sqrt{\frac{1}{n} + \frac{1}{m}}} \right] + \frac{(A_\beta - A_\alpha)}{\sqrt{\frac{1}{n} + \frac{1}{m}}}}{\sqrt{\frac{\sum_1^n (x_i - \bar{x})^2 + \sum_1^m (y_j - \bar{y})^2}{\sigma^2(m+n-2)}}}. \end{aligned}$$