A NOTE ON KAC'S DERIVATION OF THE DISTRIBUTION OF THE MEAN DEVIATION

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In a paper on a general class of estimates of deviations, Kac [3] obtained an expression for the frequency function of the estimate of mean deviation from the mean in normal samples. He was unable to establish the identity of this with an expression obtained earlier by me [1]. I now shew that the two results are, in fact, equivalent.

Kac uses the functions $\zeta^{(k)}(x)$, defined as the k — fold convolution of

$$\zeta(x) = \begin{cases} 0, & x < 0; \\ e^{-\frac{1}{2}n^2x^2}, & x \geq 0. \end{cases}$$

I used the functions $G_k(x)$ defined by the recurrence relation

$$(1) \qquad G_0(x) = 1, \qquad G_k(x) = \int_0^x e^{-(t^2/2k(k+1))}\, G_{k-1}(t)\, dt$$

Now I have shewn elsewhere [2] that the integral of $e^{-\frac{1}{2}(x_1^2+\cdots+x_k^2)}$ taken through the interior of a regular simplex in k dimensions, with its centroid at the origin and of side a, is $\sqrt{k+1}\, G_k(a/\sqrt{2})$. The relation (1) corresponds to a dissection of the simplex into sections, which are $(k - 1)$-dimensional simplexes, by joining the centroid to the vertices and taking sections parallel to the base of each of the $(k + 1)$ smaller simplexes so formed. If however we take sections parallel to a base of the whole simplex we get another recurrence relation, viz.

$$(2) \qquad G_k(x) = \int_0^x e^{-((kx-(k+1)t)^2/2k(k+1))} G_{k-1}(t)\, dt\,.$$

Now (2) may be re-written

$$\frac{G_k(nx)e^{-(n^2x^2/2(k+1))}}{n^k} = \int_0^x e^{-(n^2(x-t)^2/2)}\, \frac{G_{k-1}(nt)e^{-(n^2t^2/2k)}}{n^{k-1}}\, dt$$

whence, by induction, $G_{k-1}(nx)\cdot e^{-(n^2x^2/2k)} = n^{k-1}\zeta^{(k)}(x)$ and the equivalence of Kac's result to mine is established.

REFERENCES

[1] H. J. Godwin, "On the distribution of the estimate of mean deviation obtained from samples from a normal population," *Biometrika*, Vol. 33 (1945), pp. 254–256.

[2] H. J. Godwin, "A further note on the mean deviation," *Biometrika*, Vol. 35 (in the press).

[3] M. Kac, "On the characteristic functions of the distributions of estimates of various deviations in samples from a normal population," *Annals of Math. Stat.*, Vol. 19 (1948), pp. 257–261.