NOTES

This section is devoted to brief research and expository articles and other short items.

NOTE ON THE CONSISTENCY OF THE MAXIMUM LIKELIHOOD ESTIMATE[1]

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1. Introduction. The problem of consistency of the maximum likelihood estimate has been treated in the literature by several authors (see, for example, Doob [1][2] and Cramér [2][3]). The purpose of this note is to give another proof of the consistency of the maximum likelihood estimate which may be of interest because of its relative simplicity and because of the easy verifiability of the underlying assumptions. The present proof has some common features with that given by Doob, insofar that both proofs make no differentiability assumptions (thus, not even the existence of the likelihood equation is postulated) and both are based on the strong law of large numbers and an inequality involving the log of a random variable. The assumptions in the present note are stronger in some respects than those made by Doob, but also the results obtained here are stronger. For the sake of simplicity, the author did not attempt to give the most general results or to weaken the underlying assumptions as much as possible. Remarks on possible generalizations are made in Section 4.

Let $X_1, X_2, \cdots$, etc. be independently and identically distributed chance variables. The most frequently considered case in the literature is that where the common distribution is known, except for the values of a finite number of

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[2] According to a communication from Doob, his Theorem 4 is incorrect, but is correct if the class of almost everywhere continuous functions in that theorem is replaced by a suitable class C of functions. The class C can be any one of a variety of classes; for example, the class of bounded almost everywhere continuous functions, or the larger class of almost everywhere continuous functions each of which is less than or equal in modulus to any one of a prescribed sequence of functions with finite expectations. His Theorem 5 on the consistency of the maximum likelihood is then dependent on the class C used in Theorem 4.

[3] The proof given by Cramér [2], pp. 500–504, establishes the consistency of some root of the likelihood equation but not necessarily that of the maximum likelihood estimate when the likelihood equation has several roots. Recently, Huzurbazar [3] showed that under certain regularity conditions the likelihood equation has at most one consistent solution and that the likelihood function has a relative maximum for such a solution. Since there may be several solutions for which the likelihood function has relative maxima, Cramér's and Huzurbazar's results taken together still do not imply that a solution of the likelihood equation which makes the likelihood function an absolute maximum is necessarily consistent.