A NOTE ON SIMULTANEOUS CONFIDENCE INTERVALS

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Scheffé [1] and Bose and Roy [2] have recently given a method for making confidence interval estimates useful in the analysis of variance. Here we show that we can append to such a set of confidence interval estimates an additional one on D, where D^2/σ^2 is the "distance function" which determines the power of the analysis of variance test. For completeness, we first determine the same set of simultaneous confidence intervals as in [1] and [2] in a simple way. This may be of at least pedagogic interest.

Lemma. *Let $x_1, \cdots, x_k$ stand for a point in k-dimensional Euclidean space. The equations of the two planes normal to a vector $a_1, \cdots, a_k$ and tangent to a sphere of radius R centered at $c_1, \cdots, c_k$ are given by*

$$\sum a_i x_i = \sum a_i c_i + R\left(\sum a_i^2\right)^{1/2}, \qquad \sum a_i x_i = \sum a_i c_i - R\left(\sum a_i^2\right)^{1/2}.$$

This is an elementary geometric fact which we will not prove.

The analysis of variance problem can be formulated in terms of

$$y = (y_1, \cdots, y_n),$$

a set of n independent, normal random variables, each with variance σ^2, and of $\hat{\pi}_1(y), \cdots, \hat{\pi}_k(y)$, which are k independent random variables, each a linear function of $y_1, \cdots, y_n$. Each $\hat{\pi}_i$, for $i = 1, \cdots, k$, is the linear estimate of minimum variance of $E\hat{\pi}_i = \pi_i$; the variance of $\hat{\pi}_i$ is σ^2. Let ms^2 be independent of

$$\hat{\pi}_1, \cdots, \hat{\pi}_k$$

and distributed as the sum of squares of m independent $N(0, \sigma)$ random variables. The problem is to estimate or test hypotheses about $\pi_1, \cdots, \pi_k$ or linear combinations of them.

Let F_α be a number such that $(0, F_\alpha)$ includes $1 - \alpha$ of the probability of the F distribution with (k, m) d.f. Let C be a k-sphere centered at $\hat{\pi}_1, \cdots, \hat{\pi}_k$ with radius $(kF_\alpha s^2)^{1/2}$. For a vector $a = a_1, \cdots, a_k$, let $P_1(a)$ and $P_2(a)$ be the two planes which are normal to a and tangent to C. Then

$$1 - \alpha = \Pr\left\{\sum_{i=1}^{k} \frac{(\hat{\pi}_i - \pi_i)^2}{ks^2} \leqq F_\alpha\right\}, \tag{1}$$

which is the probability that C covers the point $\pi_1, \cdots, \pi_k$. This in turn is the probability that $\pi_1, \cdots, \pi_k$ is located between $P_1(a)$ and $P_2(a)$ for all vectors a. Thus

$$1 - \alpha = \Pr\left\{\sum a_i\hat{\pi}_i - \left(kF_\alpha \sum a_i^2 s^2\right)^{1/2} \leqq \sum a_i \pi_i \leqq \sum a_i \hat{\pi}_i + \left(kF_\alpha \sum a_i^2 s^2\right)^{1/2} \text{ for all vectors } a\right\}. \tag{2}$$

The last step follows from the lemma.

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