THE MODIFIED MEAN SQUARE SUCCESSIVE DIFFERENCE AND RELATED STATISTICS[1]

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1. Introduction. In estimating the variance of a normal population one uses the statistic $s^2 = (n-1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ because of its optimum properties. In certain cases where there is an indeterminable trend in the data, it has been thought useful to estimate the variance by another statistic, namely the mean square successive difference, the mean of the squared first differences, studied by J. von Neumann *et al.* [5], which eliminates a good deal of the trend and under some conditions is less biased than s^2. An explicit form of the exact distribution of this statistic seems, at least for the present, too difficult to obtain. However, by applying a device analogous to one used by Durbin and Watson [1], that is, by dropping from the mean square successive difference the middle term for an even number of observations and the two middle terms for the odd case, we find that the quadratic form has double roots, thus enabling us to obtain exact distributions in terms of elementary functions. In addition we define analogues of the Student t and the Fisher F using similarly modified statistics and derive their exact distributions when the observations are independent.

The results of this paper are mainly the exact distributions of these statistics and were given at the April 1955 meetings of the Institute of Mathematical Statistics. A short while after, these same results, independently derived, were published by A. R. Kamat [3]. Since Kamat has already published the exact distributions of these statistics and the motivation for them, it would be inappropriate to rederive them here; hence we shall only state the results and give that material that Kamat had not considered in his paper.

2. The modified mean square successive difference. Let x_i be $N(0, \sigma^2)$ and let $x_1, \cdots, x_{2m}$ be independent. We define the modified mean square successive difference to be

$$\delta_0^2 = 4^{-1}(m-1)^{-1} \sum_{\substack{i=1 \\ i \neq m}}^{2m-1} (x_{i+1} - x_i)^2. \tag{2.1}$$

The exact density of δ_0^2 is

$$\begin{aligned} \mathcal{P}_m(\delta_0^2) = \frac{4^{m-2}(m-1)}{m\sigma^2} \sum_{k=1}^{m-1} (-1)^{k+1} \sin^2 \frac{k\pi}{m} \\ \cdot \cos^{2m-6} \frac{k\pi}{2m} \exp\left[-\frac{(m-1)\sec^2 \frac{k\pi}{2m}}{2\sigma^2} \right] \end{aligned} \tag{2.2}$$

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