# ESTIMATES FOR GLOBAL CENTRAL LIMIT THEOREMS[1]

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**1. Introduction.** Let $\xi_1, \xi_2, \cdots$ be independent random variables having the same d.f. (distribution function) $F(x)$. Thus, for each $k = 1, 2, 3, \cdots$

$$\Pr\{\xi_k \leqq x\} = F(x), \qquad -\infty < x < \infty, \tag{1.1}$$

where $F(x)$ is a real monotone increasing function for which $F(-\infty) = 0$ and $F(\infty) = 1$. Let $\phi(t)$, defined by

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x), \qquad -\infty < t < \infty, \tag{1.2}$$

denote the c.f. (characteristic function) of $F(x)$. We suppose that

$$\int_{-\infty}^{\infty} x\, dF(x) = 0, \qquad \int_{-\infty}^{\infty} x^2\, dF(x) = 1, \tag{1.3}$$

so that $F(x)$ has mean 0 and standard deviation 1.

The d.f. $\tilde{F}_n(x)$ and the c.f. $\tilde{\phi}_n(x)$ of the sum $\xi_1 + \xi_2 + \cdots + \xi_n$ are such that $\tilde{\phi}_n(x) = [\phi(x)]^n$ and hence

$$[\phi(t)]^n = \int_{-\infty}^{\infty} e^{itx}\, d\tilde{F}_n(x). \tag{1.4}$$

The d.f. of the combination

$$\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n^{1/2}} \tag{1.5}$$

is then $\tilde{F}_n(n^{1/2}x)$ and we denote this by $F_n(x)$. It's c.f. is $[\phi(n^{-1/2}t)]^n$, that is,

$$[\phi(n^{-1/2}t)]^n = \int_{-\infty}^{\infty} e^{itx}\, dF_n(x). \tag{1.51}$$

The hypotheses (1.3) imply that the formulas (1.3) hold when $F(x)$ is replaced by $F_n(x)$. A special case of the central limit theorem asserts that, for each individual $x$ in the interval $-\infty < x < \infty$,

$$\lim_{n\to\infty} F_n(x) = \Phi(x) \tag{1.6}$$

where $\Phi(x)$ is the Gaussian d.f. defined by

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-u^2/2}\, du. \tag{1.61}$$

For an exposition of the above facts, see Cramér [2].

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