PÓLYA TYPE DISTRIBUTIONS OF CONVOLUTIONS[1]

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1. Introduction. The theory of totally positive kernels and Pólya type distributions has been decisively and extensively applied in several domains of mathematics, statistics, economics and mechanics. Totally positive kernels arise naturally in developing procedures for inverting, by differential polynomial operators [7], integral transformations defined in terms of convolution kernels. The theory of Pólya type distributions is fundamental in permitting characterizations of best statistical procedures for decision problems [8] [9] [13]. In clarifying the structure of stochastic processes with continuous path functions we encounter totally positive kernels [11] [12]. Studies in the stability of certain models in mathematical economics frequently use properties of totally positive kernels [10]. The theory of vibrations of certain types of mechanical systems (primarily coupled systems) involves aspects of the theory of totally positive kernels [5].

In this paper, we characterize new classes of totally positive kernels that arise from summing independent random variables and forming related first passage time distributions.

A function $f(x, y)$ of two real variables ranging over linearly ordered one dimensional sets X and Y respectively, is said to be *totally positive of order* k (TP_k) if for all $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$, $(x_i \varepsilon X; y_j \varepsilon Y)$ and all $1 \leqq m \leqq k$,

$$(1) \qquad f\begin{pmatrix} x_1, x_2, \cdots, x_m \\ y_1, y_2, \cdots, y_m \end{pmatrix} = \begin{vmatrix} f(x_1, y_1) & f(x_1, y_2) & \cdots & f(x_1, y_m) \\ f(x_2, y_1) & f(x_2, y_2) & \cdots & f(x_2, y_m) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_m, y_1) & f(x_m, y_2) & \cdots & f(x_m, y_m) \end{vmatrix} \geqq 0.$$

Typically, X is an interval of the real line, or a countable set of discrete values on the real line such as the set of all integers or the set of non-negative integers; similarly for Y. When X or Y is a set of integers, we may use the term "sequence" rather than "function."

A related, weaker property is that of sign regularity. A function $f(x, y)$ *is sign regular of order* k, if for every $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$, and $1 \leqq m \leqq k$, the sign of

$$f\begin{pmatrix} x_1, x_2, \cdots, x_m \\ y_1, y_2, \cdots, y_m \end{pmatrix}$$

depends on m alone.

If a TP_k function $f(x, y)$ is a probability density in one of the variables, say x, with respect to a σ-finite measure $\mu(x)$, for each fixed value of y, then $f(x, y)$

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