SOME ASYMPTOTIC RESULTS FOR A COVERAGE PROBLEM

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1. Introduction. A quantity whose distribution is of considerable interest in calculation of microscopic behavior of heterogeneous materials is the intercept fraction of the phases of the mixture (i.e., the fraction of a linear path intercepted by a particular phase). For example, in nuclear reactor theory, one will be interested in the fraction of a neutron path through a given phase. In this paper, we study the statistical behavior of the intercept fraction, for a path of fixed length, under the following idealization (more precisely defined in Section 2):

1. Linear sections of a phase are selected at random and placed on a very long line at random, without overlap.
2. The given path length is placed at random on the long line.

Some related experimental work has been done [1], with photomicrographs of sections on solid Boron Carbide-Zirconium mixtures. In this work a number of linear paths of fixed length and parallel to one axis of the photograph were taken at positions along the other axis of the photograph and the length of Boron Carbide covered by each path was measured. It is interesting to note that the frequency of zero fraction intercept predicted by the idealization was in good agreement with experimental results. The small differences found were in the direction suggested by the fact that the sampled line had to be of finite thickness. That is, the predicted frequency of zero intercept tended to be slightly higher than observed.

2. Assumptions and Summary of Results. We assume we have a sample of line segments $\Delta_1, \Delta_2, \cdots, \Delta_n$ which are independent random drawings from a universe of segments with probability density $p(\Delta)$, $0 \leqq \Delta \leqq \Delta_M$. Now we suppose that the segments $\Delta_1, \Delta_2, \cdots, \Delta_n$ are placed on the interval $(0, L)$ in such a way that all admissible configurations of the segments are equally likely and that $L \geqq n\Delta_M$. We call a configuration admissible if

(a) There is no overlapping among segments.

(b) No segments overlap zero or L.

Now we consider a line of length $\lambda \ll L$ and place it at random on $(0, L)$ with the restriction that no overlap with zero or L occurs.

Finally we define the intercept (or coverage) of λ, λF, as that part of λ covered by the segments $\Delta_1, \Delta_2, \cdots, \Delta_n$, and ask for the distribution of λF. In particular, we consider the limiting distribution of λF as $n \to \infty$, for $n\mu/L = V$, $0 < V < 1$, where μ is average segment size.

We find that

$$\text{I. } \lim_{n\to\infty} \Pr\{\lambda F = 0\} = (1 - V) \exp - \alpha\lambda, \qquad \text{where } \alpha = V/(1 - V)\mu.$$

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