ON DEVIATIONS OF THE SAMPLE MEAN

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1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables. Let a be a constant, $-\infty < a < \infty$, and for each $n = 1, 2, \cdots$ let

$$p_n = P\left(\frac{X_1 + \cdots + X_n}{n} \geqq a\right). \tag{1}$$

It is assumed throughout the paper that the distribution of X_1 and the given constant a satisfy the conditions stated in the following paragraph. These conditions imply that $p_n > 0$ for each n, and that $p_n \to 0$ as $n \to \infty$. The object of the paper is to obtain an estimate of p_n, say q_n, which is precise in the sense that

$$q_n/p_n = 1 + o(1) \quad \text{as} \quad n \to \infty. \tag{2}$$

Let t be a real variable, and let $\varphi(t)$ denote the moment generating function (m.g.f.) of X_1, i.e., $\varphi(t) = E(e^{tX_1})$, $0 < \varphi \leqq \infty$. Define

$$\psi(t) = e^{-at}\varphi(t). \tag{3}$$

Let T denote the set of all values t for which $\varphi(t) < \infty$. *We suppose that* $P(X_1 = a) \neq 1$, *that* T *is a non-degenerate interval, and that there exists a positive* τ *in the interior of* T *such that* $\psi(\tau) = \inf_t \{\psi(t)\} = \rho$ (say). These conditions are satisfied if, for example, $\varphi(t) < \infty$ for all t, $E(X_1) = 0$, $a > 0$, and $P(X_1 > a) > 0$. In any case, τ and ρ are uniquely determined by

$$\frac{\varphi'(\tau)}{\varphi(\tau)} = a \quad \text{and} \quad \rho = \psi(\tau), \tag{4}$$

where $\varphi' = d\varphi/dt$, and we have $0 < \rho < 1$.

There are three separate cases to be considered.

Case 1: The distribution function (d.f.) of X_1 is absolutely continuous, or, more generally, this d.f. satisfies Cramér's condition (C) [1, p. 81].

Case 2: X_1 is a lattice variable, i.e., there exist constants x_0 and $d > 0$ such that X_1 is confined to the set $\{x_0 + rd : r = 0, \pm 1, \pm 2, \cdots\}$ with probability one.

Case 3: Neither Case 1 nor Case 2 obtains.

We can now state

THEOREM 1. *There exists a sequence* $b_1, b_2, \cdots$ *of positive numbers* b_n *such that*

$$p_n = \frac{\rho^n}{(2\pi n)^{\frac{1}{2}}} b_n [1 + o(1)], \qquad \log b_n = O(1) \tag{5}$$

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