A "RENEWAL" LIMIT THEOREM FOR GENERAL STOCHASTIC PROCESSES[1]

By V. E. Beneš

Bell Telephone Laboratories and Dartmouth College

1. Introduction. Let $\{x_t, -\infty < t < \infty\}$ be a stochastic process that takes values in a (possibly abstract) space X. Specifically, assume as given a space Ω of points ω, a Borel field $\mathfrak{F}$ of ω-sets, a probability measure P of $\mathfrak{F}$-sets, a Borel field $\mathfrak{A}$ of X-sets, and for each real t a function $x_t(\cdot)$ from Ω to X such that $\{x_t \varepsilon A\}$ is an $\mathfrak{F}$-set for $A \varepsilon \mathfrak{A}$. The existence and calculation of

$$\lim_{t\to\infty} \Pr\{x_t \varepsilon A\}, \tag{1}$$

under the weakest possible conditions, is a problem of considerable interest in probability theory.

Some processes for which this problem has received much attention are the Markov, semi-Markov, and regenerative processes. (The term 'semi-Markov' is Lévy's in [6], while the more inclusive 'regenerative' is Smith's, in [10] and [11].) These random processes share the very strong property of having *regeneration points*, junctures at which the previous history (except for the present state, possibly) of the process becomes irrelevant to its future development. In Smith's phrase, the past loses all prognostic significance at a regeneration point. In a sense, then, the process consists of segments that are mutually independent. (See Smith's construction, [11], p. 256.) Because of this independence, the renewal theorems of Feller [4], Blackwell [1], and Smith [9] have been directly useful in studying the limit (1) for these processes.

Our limit theorem is of "renewal" type in the sense that it depends on a relationship between x_t and a discrete parameter real process $\{S_k, k \text{ an integer}\}$ that is analogous to the classical (sequence of) renewal points. However, we do not assume that any *regeneration* occurs at these points. Our theorem differs from previous results in requiring no assumptions of independence, and resting only on mild stationarity properties, to the effect that certain kernels are difference kernels. The approach that we use to study (1) amounts to carefully distinguishing between the *independence* and the stationarity properties of x_t, and using only the latter. The approach is fruitful because (as will appear) our theory applies to many processes that do not have regeneration points.

The present paper was inspired in large part by the work of W. L. Smith (on renewal theory and regenerative processes); he will recognize in it generalizations and copies of many of his ideas. A detailed discussion of the relation of our work to Smith's is given in Section 4. Also, this paper is a general develop-

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