ON A GENERALIZATION OF THE FINITE ARCSINE LAW[1]

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1. Introduction. Throughout this paper $\{X_k\}$ will denote a sequence of independent, identically distributed random variables with continuous and symmetric distributions.

Among the neatest and most startling results concerning the behavior of the partial sums $S_n = X_1 + \cdots + X_n (S_0 = 0)$ are those which are *distribution free*, i.e., those which do not depend on the distribution of X_1. For example, if we define

$$(1)\qquad \begin{aligned} &N_n : \text{the number of positive sums among } S_1, S_2, \cdots, S_n. \\ &L_n : \text{the smallest } k(=0, 1, \cdots, n) \text{ for which } S_k = \max_{0 \leqq j \leqq n} S_j, \end{aligned}$$

then Sparre Andersen [3, 4] showed that N_n and L_n have a common distribution which does not depend on the distribution of X_1:

$$(2)\qquad P\{N_n = m\} = P\{L_n = m\} = \binom{2m}{m}\binom{2n-2m}{n-m}(1/2^{2n}), \qquad 0 \leqq m \leqq n.$$

We give here another distribution free result which generalizes (2) and which includes in particular information about the *joint* distribution of N_n and L_n. It is disappointingly easy to construct examples (even for $n = 3$) to show that the total joint distribution of N_n and L_n is *not* distribution free. Yet, for the special case $L_n = n$ we can find explicitly the distribution of N_n, namely

$$(3)\qquad P\{N_n = m, L_n = n\} = (1/2n)\binom{2n-2m}{n-m}(1/2^{2n-2m}), \qquad 1 \leqq m \leqq n.$$

Our method consists of finding a pair of "differential equations" for the generating functions of quantities like those appearing in (3). These equations are then solved and the generating functions inverted.

Before we can state our main result we must introduce more notation. Let $R_{n0} \geqq R_{n1} \geqq \cdots \geqq R_{nn}$ be an ordering of the partial sums $S_0, S_1, \cdots, S_n$. Since the distribution of X_1 is continuous, the probability that two S_k's are equal is zero. This means that with probability one there is a unique index m such that $R_{nk} = S_m$. We say $L_{nk} = m$ in case $R_{nk} = S_m$, and we note that L_{nk} is well defined with probability one. Darling [2] found the distribution of L_{nk} in terms of products of binomial coefficients, but he gave no results for joint distributions.

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