STOCHASTIC PROCESSES ON A SPHERE[1]

By Richard H. Jones[2]

University of Stockholm

0. Summary. Spectral representations of stochastic fields on a sphere are given for spherically symmetric and axially symmetric cases. Unbiased estimates of the spectral parameters are presented and the variance calculated for normally distributed fields. Time varying processes are discussed briefly with reference to terrestrial situations.

1. Introduction. Let $\xi(P)$ be a real valued random field on a unit sphere, S_2, of the three-dimensional space R_3, which has finite variance and realizations, $x(P)$, which are quadratically integrable over the surface of the sphere

$$\int_P x^2(P)\, d\Omega_P < \infty;$$

Ω_P denoting the surface element, and $\int_P$ integration over the entire surface. Because of the completeness of the spherical harmonics $Y_\nu^m(P)$, the random field may be represented

$$\xi(P) = \sum_{\nu=0}^{\infty} \sum_{m=-\nu}^{\nu} Z_{\nu m} Y_\nu^m(P). \tag{1}$$

By an isotropic field we mean one whose covariance depends only on the spherical distance between two points and whose mean $E\xi(P)$ is constant. Without loss of generality, it will be assumed that $E\xi(P) = 0$, which implies $EZ_{\nu m} = 0$. The conditions for isotropy for normalized spherical harmonics

$$\left(\int_P [Y_\nu^m(P)]^2 d\Omega_P = 1\right)$$

are

$$EZ_{\nu m} Z_{\mu n} = \delta_{\nu\mu}\delta_{mn} f_\nu \geqq 0 \tag{2}$$

δ_{ij} being the Kronecker delta. The representation (1) and conditions (2) were given by Obukhov in 1947 (Yaglom, [3]).
The covariance function is

$$r(P, Q) = E\xi(P)\xi(Q) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \sum_{m=-\nu}^{\nu} \sum_{n=-\mu}^{\mu} EZ_{\nu m} Z_{\mu n} Y_\nu^m(P) Y_\mu^n(Q).$$

Calling the angle between two points P and Q, γ_{PQ}, and using the conditions for isotropy (2), $r(\gamma_{PQ}) = \sum_{\nu=0}^{\infty} f_\nu \sum_{m=-\nu}^{\nu} Y_\nu^m(P) Y_\nu^m(Q)$, which by the addition

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[2] Now at The Johns Hopkins University.