APPROXIMATIONS TO MULTIVARIATE NORMAL ORTHANT PROBABILITIES[1]

By Ralph Hoyt Bacon

GPL Division—General Precision, Inc.[2]

1. Summary. In this work, a close approximation to the positive normal orthant is obtained for the special case in which all the correlation coefficients are equal. From this expression, an approximation for the general case is suggested. For special cases, even closer approximations are obtainable.

2. Statement of the problem. Consider n correlated random variables, distributed with zero means and unit variance according to a multivariate normal distribution, whose density function we will write as $\phi_n(x_1, x_2, \cdots x_n)$. Then the probability that all n random variables will be positive is

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \phi_n(x_1, x_2, \cdots x_n)\, dx_1\, dx_2 \cdots dx_n .$$

This probability will be denoted by $P_n(c_{11}, c_{12}, \cdots c_{nn})$, where c_{11}, etc., denote the elements of the inverse of the covariance matrix. Let the correlation coefficients be denoted by ρ_{ij}. Then, since the density function obeys the relation

$$\frac{\partial \phi_n}{\partial \rho_{ij}} = \frac{\partial^2 \phi_n}{\partial x_i\, \partial x_j},$$

we have, for example,

$$\begin{aligned} \frac{\partial P_n}{\partial \rho_{12}} &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left[\int_0^\infty \int_0^\infty \frac{\partial^2 \phi_n}{\partial x_1\, \partial x_2}\, dx_1\, dx_2 \right] dx_3 \cdots dx_n \\ &= \frac{1}{2\pi(1-\rho_{12}^2)^{\frac{1}{2}}} P_{n-2}(c_{33}, c_{34}, \cdots c_{nn}). \end{aligned} \tag{1}$$

There will be $\frac{1}{2}n(n-1)$ such equations—one for each correlation coefficient. Equation (1) is a special case of Equation (6) given by Plackett [5].

3. Equal correlation coefficients. Let the resultant probability in the special case in which all the correlation coefficients are equal to be denoted by $P_n(\rho)$. Then the $\frac{1}{2}n\ (n-1)$ Equations (1) add up to

$$\frac{dP_n(\rho)}{d\rho} = \frac{n(n-1)}{4\pi(1-\rho^2)^{\frac{1}{2}}} P_{n-2}\left(\frac{\rho}{1+2\rho}\right).$$

Therefore, since $P_n(0) = (\frac{1}{2})^n$, we have

$$P_n(\rho) = \left(\frac{1}{2}\right)^n + \frac{n(n-1)}{4\pi} \int_0^\rho P_{n-2}\left(\frac{\lambda}{1+2\lambda}\right) \frac{d\lambda}{(1-\lambda^2)^{\frac{1}{2}}}. \tag{2}$$

Equation (2) is the same as Equation (102′) given by Ruben [6].

Received May 25, 1961; revised September 11, 1962.

[1] Some of these results were presented at the Atlantic City Meeting, September, 1957 [2].

[2] Formerly General Precision Laboratory.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.*
www.jstor.org