COMPARISON OF THE VARIANCE OF MINIMUM VARIANCE AND WEIGHTED LEAST SQUARES REGRESSION COEFFICIENTS[1]

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0. Introduction. This paper compares the variance and generalized variance of minimum variance (MV) and weighted least squares (WLS) estimates of regression coefficients. Matrix inequalities originally developed to study the rate of convergence of various iterative methods of solving linear equations (cf. [4]) are used in making the comparisons. These inequalities are given in Section 1 and applied in Section 2. In Section 3, attention is focused on diagonal weight matrices, and an example is given in Section 4.

1. Matrix inequalities. Let A be a real positive definite matrix with

$$Ae_i = \lambda_i e_i \qquad i = 1, 2, \cdots, n$$

$$\|e_i\| = 1$$

and the eigenvalues λ_i satisfying $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n > 0$, and let $\kappa = \lambda_1/\lambda_n$ be the *spectral condition number* of A. For any non-null vector x, define

$$\mu_k = \mu_k(x) = x'A^kx. \tag{1}$$

Then

$$1 \leqq \mu_{k+1}\mu_{k-1}/\mu_k^2 \leqq [(\kappa^{\frac{1}{2}} + \kappa^{-\frac{1}{2}})/2]^2. \tag{2}$$

For $k = 0$, the inequality on the right is the Kantorovich inequality [5], and equality is attained for $x = a(e_1 \pm e_n)(a \neq 0)$. This inequality was first derived in order to determine the rate of convergence of the method of steepest descent for solving linear equations [5]. Equality on the left is attained when $x = be_i$ ($b \neq 0, i = 1, 2, \cdots, n$).

Inequalities (2) can be generalized. Let

$$M_k = X'A^kX \quad \text{and} \quad \mu_k(X) = \det M_k. \tag{3}$$

If X has rank p, define the condition number by

$$\kappa_p = \lambda_1 \cdots \lambda_p/\lambda_{n-p+1} \cdots \lambda_n. \tag{4}$$

Inequality (2) becomes

$$1 \leqq \mu_{k+1}(X)\cdot\mu_{k-1}(X)/\mu_k^2(X) \leqq [(\kappa_p^{\frac{1}{2}} + \kappa_p^{-\frac{1}{2}})/2]^2. \tag{5}$$

Proofs for inequalities (2) and (5) are given by Schopf [8].

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