MINIMAX THEOREMS ON CONDITIONALLY COMPACT SETS

By Teh Tjoe-Tie

Catholic University of Louvain

1. Introduction. Conditionally compact sets in minimax theorems were first considered by A. Wald [2]: Let $K(x, y)$ be a real-valued bounded function defined on the product of two arbitrary sets X and Y. The distances

$$d(x_1, x_2) = \sup_Y |K(x_1, y) - K(x_2, y)| \quad \text{for } x_1, x_2 \varepsilon X$$

$$d(y_1, y_2) = \sup_X |K(x, y_1) - K(x, y_2)| \quad \text{for } y_1, y_2 \varepsilon Y$$

define metric topologies for X and Y respectively which will be referred tos a the *intrinsic* topologies or, briefly, the (I)-*topologies* for X and Y with respect to the function K. In general these topologies are pseudo-metric only, but we assume that a reduction to equivalent classes has made them properly metric.

Now let P be the set of all probability measures p on $\mathcal{I}_X$, i.e. the σ-algebra, generated by the (I)-open sets in X. Similarly, Q is the set of all probability measures q on $\mathcal{I}_Y$, the σ-algebra, generated by the (I)-open sets in Y. Then, if $K(p, q) = \int K(x, y)\, dp(x) \times dq(y)$, we have [2]:

Theorem 1.1. *If one of the spaces X and Y is (I)-conditionally compact, then both spaces are (I)-conditionally compact and* $\sup_P \inf_Q K(p, q) = \inf_Q \sup_P K(p, q)$.

A metric space is said to be conditionally compact if and only if, given any $\epsilon > 0$, there exists a finite subset $\{x_1, \cdots, x_n\}$ of X such that the class of spheres $S(x_i, \epsilon) = \{x : d(x, x_i) \leqq \epsilon\}$ $(i = 1, \cdots, n)$ is a covering for X.

A concept which is equivalent to (I)-conditionally compactness is that of almost periodic functions defined as follows [1]: A real-valued bounded function $K(p, q)$ defined on the product of two sets P and Q is *left almost periodic* if and only if, given $\epsilon > 0$, there exists a finite subset $\{p_1, \cdots, p_n\}$ of P such that for any $p \varepsilon P$ there is some p_i, $1 \leqq i \leqq n$, for which $|K(p, q) - K(p_i, q)| \leqq \epsilon$, for all $q \varepsilon Q$.

An analogous definition holds for *right almost periodicity*. Obviously, right almost periodicity follows from left almost periodicity, and vice versa.

The following definitions are due to Ky Fan [1]: A real-valued function $K(p, q)$ is said to be *concave-like* in p if and only if, given any $t \varepsilon [0, 1]$ and any $p_1, p_2 \varepsilon P$, there exists $p_0 \varepsilon P$ such that the inequality $tK(p_1, q) + (1 - t)K(p_2, q) \leqq K(p_0, q)$ holds for all $q \varepsilon Q$.

$K(p, q)$ is said to be *convex-like* in q if and only if, given any $t \varepsilon [0, 1]$ and any $q_1, q_2 \varepsilon Q$ there exists $q_0 \varepsilon Q$ such that the inequality $tK(p, q_1) + (1 - t)K(p, q_2) \geqq K(p, q_0)$ holds for all $p \varepsilon P$.

$K(p, q)$ is *concave-convex-like* if it is concave-like in p and convex-like in q.

Received December 17, 1962; revised June 11, 1963.