NONPARAMETRIC CONFIDENCE INTERVALS FOR A SHIFT PARAMETER[1]

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Summary. Exact expressions and large-sample approximations are given for the nonparametric confidence intervals for a shift parameter Δ, which are obtained from the two-sample Wilcoxon test. These intervals are shown to have the same asymptotic efficiency relative to the standard confidence intervals for Δ as the Wilcoxon test has relative to Student's t-test. As a consequence of this result, a constant multiple of the length of the nonparametric intervals is shown to be a consistent estimator of the quantity $1/\int f^2(x)\,dx$.

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be independent observations from distributions $P[X_i \leqq x] = F(x)$ and $P[Y_j \leqq y] = F(y - \Delta)$ respectively, where F is assumed to be continuous but otherwise unknown. Exact confidence intervals for Δ can be based on nonparametric tests such as the Wilcoxon test. Let $U(\Delta)$ be the number of differences $Y_j - X_i$ which exceed Δ, and suppose that the two-sided symmetric level α test of the hypothesis $H(\Delta_0) : \Delta = \Delta_0$ accepts $H(\Delta_0)$ when

$$C_\alpha \leqq U(\Delta_0) \leqq mn - C_\alpha . \tag{1}$$

If the ordered set of mn differences $Y_j - X_i$ is denoted by

$$D^{(1)} < \cdots < D^{(mn)},$$

the confidence intervals obtained from (1) by solving for Δ_0 and replacing Δ_0 by Δ are seen to be

$$\Delta_L < \Delta < \Delta_U \tag{2}$$

where

$$\Delta_L = D^{(C_\alpha)} \quad \text{and} \quad \Delta_U = D^{(mn+1-C_\alpha)}. \tag{3}$$

For given sample sizes m and n, a constant C_α for which (1) has exactly probability $1 - \alpha$ may not exist. For the large-sample problems to be considered below it is enough that the constants $C_\alpha = C_\alpha(m, n)$ are chosen in such a way that the probability $1 - \alpha(m, n)$ of (1) tends to the specified value $1 - \alpha$ as m and n tend to infinity.

For small and moderate values of m and n, the critical value C_α can be read from tables of the null distribution of the Wilcoxon statistic. For large m and n, one has the following approximation.

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