BALANCED DESIGNS WITH UNEQUAL NUMBERS OF REPLICATES

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1. Introduction. Rao [5] proved as a theorem that a necessary and sufficient condition for a design to be balanced, i.e., for $\text{var}(\hat{\tau}_i - \hat{\tau}_{i'})$ to be the same for all pairs (i, i'), $i \neq i'$, is that the matrix $\mathbf{C}$ of the adjusted intrablock normal equations shall have all its diagonal elements equal and all its off-diagonal elements equal. If r_i denotes the number of times that the ith treatment is replicated, k_j denotes the number of plots in the jth block and $\mathbf{n} = (n_{ij})$ is the incidence matrix, the elements of $\mathbf{C}$ are

$$c_{ii} = r_i - \sum_j (n_{ij}^2/k_j); \qquad c_{ii'} = -\sum_j (n_{ij}n_{i'j}/k_j), \qquad i \neq i'.$$

The design is said to be proper if $k_j = k$ for all k, equireplicate if all the r_i are equal and binary if n_{ij} takes only the values 0 or 1.

Rao also proved as a corollary that if a binary balanced design is proper, then it must be equireplicate, and gave an example of an equireplicate binary balanced design with unequal block sizes. Results published later by Atiqullah [1], Graybill [3], and Hanani [4], enable us to give a simpler proof of Rao's theorem and to derive examples showing that the corollary does not hold when either the binary requirement or the requirement of equal block sizes is relaxed.

2. The theorem. $\mathbf{1}_v$ will denote the column vector of v elements each of which is unity; $\mathbf{J}_v$ will denote the matrix $\mathbf{1}_v\mathbf{1}_v'$. $\mathbf{C}$ is a real symmetric matrix of rank $(v - 1)$ such that $\mathbf{C1} = 0$. The normal equations are solved under the side condition $\mathbf{1}'\boldsymbol{\tau} = \mathbf{1}'\hat{\boldsymbol{\tau}} = 0$.

Graybill's method of solution ([3], p. 292) is to consider the augmented matrix $\mathbf{C}^*$ and its inverse

$$\mathbf{C}^* = \begin{pmatrix} \mathbf{C} & \mathbf{1} \\ \mathbf{1}' & 0 \end{pmatrix}, \mathbf{C}^{*-1} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & B_{22} \end{pmatrix}.$$

Then it is readily shown that

$$\hat{\boldsymbol{\tau}} = \mathbf{B}_{11}\mathbf{Q} \tag{1}$$

$$\mathbf{1}'\mathbf{B}_{11} = 0 = \mathbf{B}_{11}\mathbf{1} \tag{2}$$

$$\text{cov}\,(\hat{\boldsymbol{\tau}}) = \mathbf{B}_{11}\mathbf{C}\mathbf{B}_{11}\sigma^2 = \mathbf{B}_{11}\sigma^2 \tag{3}$$

$$\mathbf{C}\mathbf{B}_{11} = \mathbf{I}_v - v^{-1}\mathbf{J}_v = \mathbf{B}_{11}\mathbf{C}. \tag{4}$$

Atiqullah [1] has shown that necessary and sufficient conditions for a design to be balanced are that var $\hat{\tau}_i$ shall be the same for all i, and that cov $(\hat{\tau}_i, \hat{\tau}_{i'})$ shall be the same for all pairs i, i', $i \neq i'$. From (2) and (3) these conditions are equivalent to $\mathbf{B}_{11} = a(\mathbf{I}_v - v^{-1}\mathbf{J}_v)$ where a is some positive constant.

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