ON THE TOPOLOGICAL STRUCTURE OF SOME ORDERED FAMILIES OF DISTRIBUTIONS[1]

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1. Introduction. In many statistical contexts one studies a family $\mathfrak{F}$ of distribution functions F on the real line with the following properties. (a) $\mathfrak{F}$ is a dominated family; (b) $\mathfrak{F} = \{F_\theta\}$ where θ is a real parameter taking values in an interval; (c) convergence of a sequence in $\mathfrak{F}$ to an element of $\mathfrak{F}$ in the usual (weak) sense is represented by the convergence of the corresponding parameter points; (d) the natural order of θ values corresponds to a simple ordering of the elements of $\mathfrak{F}$ in terms of probability.

An example of such an ordering in terms of probability is $\theta_1 \geqq \theta_2 \Leftrightarrow F_{\theta_1}(t) \leqq F_{\theta_2}(t)$ for all real t, this ordering of $\mathfrak{F}$ being typical of cases where θ is a location parameter. An example of (d), with θ a scale parameter is F_θ = the normal distribution function with mean zero and variance θ; here $\theta_1 \geqq \theta_2 \Leftrightarrow G_1(s) \leqq G_2(s)$ *for all real* s, where G_θ is the distribution function of the sufficient statistic $s = t^2$; in view of the one-one correspondence through θ, between the families $\{F_\theta\}$ and $\{G_\theta\}$, the probabilistic ordering of $\{G_\theta\}$ corresponds to such an ordering for $\{F_\theta\}$.

The main object of this paper is to study an arbitrary set $\mathfrak{F}$ of distribution functions which is simply ordered according to the following slightly stronger order relationship: $\theta_1 \geqq \theta_2 \Leftrightarrow F_{\theta_1}(t)/F_{\theta_2}(t)$ and $[1 - F_{\theta_1}(t)]/[1 - F_{\theta_2}(t)]$ are nondecreasing functions of t. This order relation is still weaker than the natural order relation of monotone likelihood ratio families. It is shown, that for such ordered families $\mathfrak{F}$, convergence in the weak sense is equivalent to convergence in the strong sense (see Theorem 1). Thus, we obtain for ordered families much stronger results than can be obtained in the general case. In general, conclusions from weak convergence to a stronger type of convergence can be drawn only if the limit function is continuous (see R. R. Rao (1962) p. 662, Theorem 3.1).

Furthermore it is shown that (except for certain pathological situations) the order relationship defined above implies that $\mathfrak{F}$ is a one-parameter family for which (a), (b) and (c) hold (see Theorems 2, 3 and 4).

In Section 2 the relevant notions of order and distances are defined and discussed. The main theorems are stated in Section 3 and proved in Section 5 using the lemmata of Section 4. In Section 6 it is pointed out that the present results extend to probability measures on arbitrary spaces provided there exists a

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